British Journal of Mathematics & Computer Science
4(9): 1202-1212, 2014

SCIENCEDOMAIN *international*
www.sciencedomain.org

A Weaker Condition for Stable Lagrange Duality in DC Conic Programming

Mengdan Wang[1] **and Donghui Fang**[2*]

[1] *College of Mathematics and Statistics, Jishou University, Jishou 416000, P. R. China*

Original Research Article

Received: 26 December 2013
Accepted: 09 February 2014
Published: 25 February 2014

Abstract

We consider the DC conic programming in locally convex Hausdorff vector spaces. By using the infimal convolution of the conjugate functions, we present a new regularity condition, which turns out to be weaker than the regularity conditions given so far in the literature. Moreover, it provides a sufficient and necessary condition for the stable Lagrange duality for the DC conic programming.

Keywords: regular condition; Lagrange duality; Fenchel-Lagrange duality; DC conic programming
2010 Mathematics Subject Classification: 90C26;90C46

1 Introduction

Let X and Y be real locally convex Hausdorff vector spaces and $C \subseteq X$ be a nonempty convex set. Let $S \subseteq Y$ be a closed convex cone and $S^{\oplus}$ the positive dual cone of S. Let $f, g : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be proper functions and $h : X \to Y$ be an S-convex mapping with respect to the cone S. Consider the following DC conical programming:

$$(P) \qquad \begin{array}{ll} \text{Min} & f(x) - g(x), \\ \text{s. t.} & x \in C,\ h(x) \in -S. \end{array}$$

This problem has been studied in [1],[2],[3],[4],[5] and also studied in [6],[7],[8],[9] for the special case when $g = 0$. Following [1], the Lagrange dual problem of (P) is defined by

$$(D) \quad \inf_{u^* \in \operatorname{dom} g^*} \sup_{\lambda \in S^{\oplus}} \{g^*(u^*) - (f + \delta_C + \lambda h)^*(u^*)\}.$$

Let $v(P)$ and $v(D)$ denote the optimal values of problems (P) and (D), respectively. Obviously, if g is lower semicontinuous, then problems (P) and (D) satisfy the so-called weak duality, i.e., $v(p) \geq v(D)$, but a duality gap may occur, that is, we may have $v(P) > v(D)$. A challenge has been to give sufficient conditions which guarantee the Lagrange duality, the situation when $v(P) = v(D)$. As

*Corresponding author: E-mail: dh_fang@163.com

mentioned in [10], the question of finding condition, which ensures the Lagrange duality, is not only important for understanding the fundamental feature of convex programming but also for the efficient development of numerical schemes.

Over the years, numerous conditions have been given in the literature ensuring the Lagrange duality for the special case when $g = 0$ (cf. [10],[11],[12]). In particular, the authors in [10] presented some constraint qualifications which completely characterize the Lagrange duality for convex programming problems in Banach spaces and they established necessary and sufficient dual conditions for the stable Lagrange duality in [12] under the assumptions that $C = X$ and h is continuous. For the case when $g \neq 0$, some duality results for problems (P) and (D) have been established in many paper, (see [1],[4] and the references therein). For example, Dinh et.al. established in [4] the strong Lagrange duality via a closedness condition and Fang presented in [1] a new constraint qualification which completely characterizes the stable Lagrange duality for problems (P) and (D).

In this paper, we continuous to study the Lagrange duality for problems (P) and (D). We first introduce a new regularity condition, which is formulated by using the infimal convolution of the conjugate functions and turns out to be weaker than the conditions that has been given in [1]. Moreover, it generalizes the condition introduced by Jeyakumar and Li in [10],[12]. Then we succeed to obtain a sufficient and necessary condition which guarantees the Lagrange duality between problem (P) and its Lagrange dual problem.

The rest of the paper is organized as follows. In Section 2 we present basic notations and preliminary results. A weaker regularity condition is introduced and the stable Lagrange duality between (P) and (D) is considered in Section 3.

2 Notations and preliminary results.

The notations used in the present paper are standard (cf. [13]). In particular, we assume throughout the whole paper that X and Y are real locally convex Hausdorff topological vector spaces, and let X^* denote the dual space of X, endowed with the weak*-topology $w^*(X^*, X)$. By $\langle x^*, x\rangle$, we shall denote the value of the functional $x^* \in X^*$ at $x \in X$; i.e., $\langle x^*, x\rangle = x^*(x)$. Let Z be a set in X. The closure and interior of Z are denoted by $\mathrm{cl}\, Z$ and $\mathrm{int}\, Z$, respectively. Thus if $W \subseteq X^*$, then $\mathrm{cl}\, W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The indicator function δ_A of the nonempty set A is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $f : X \to \bar{\mathbb{R}}$ be a proper convex function. The effective domain, the epigraph and the conjugate function of f are denoted by $\mathrm{dom}\, f, \mathrm{epi}\, f$ and f^*, respectively; they are defined by

$$\mathrm{dom}\, f := \{x \in X : f(x) < +\infty\},$$

$$\mathrm{epi}\, f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\},$$

and

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\} \quad \text{for each } x^* \in X^*.$$

It is well known and easy to verify that $\mathrm{epi}\, f^*$ is weak*-closed. The lsc hull of f, denoted by $\mathrm{cl}\, f$, is defined by $\mathrm{epi}\,(\mathrm{cl}\, f) = \mathrm{cl}\,(\mathrm{epi}\, f)$. Then $f^* = (\mathrm{cl}\, f)^*$ (cf. [13]). If $\mathrm{cl}\, f$ is proper and convex, then $f^{**} = \mathrm{cl}\, f$. By definition, the Young-Fenchel inequality below holds:

$$f(x) + f^*(x^*) \ge \langle x, x^*\rangle \quad \text{for each pair } (x, x^*) \in X \times X^*. \tag{2.1}$$

In particular, let $p \in X^*$. Define a function on X that $p(x) := \langle p, x\rangle, \text{ for each } x \in X$. Then, for any $a \in \mathbb{R}$ and any function $h : X \to \bar{\mathbb{R}}$,

$$(h + p + a)^*(x^*) = h^*(x^* - p) - a \quad \text{for each } x^* \in X^*;$$

$$\operatorname{epi}(h + p + a)^* = \operatorname{epi} h^* + (p, -a).$$

If g, h are proper, then

$$\operatorname{epi} g^* + \operatorname{epi} h^* \subseteq \operatorname{epi}(g + h)^*, \tag{2.2}$$

$$g \le h \Rightarrow g^* \ge h^* \Leftrightarrow \operatorname{epi} g^* \subseteq \operatorname{epi} h^*. \tag{2.3}$$

Moreover, if g is convex and lsc on $\operatorname{dom} h$, then, by [14, Lemma 2.3],

$$\operatorname{epi}(h - g)^* = \bigcap_{x^* \in \operatorname{dom} g^*} (\operatorname{epi} h^* - (x^*, g^*(x^*))). \tag{2.4}$$

Finally, we define the infimal convolution of g and h as the function $g \Box h : X \to \mathbb{R} \cup \{\pm\infty\}$ given by

$$(g \Box h)(a) := \inf_{x \in X} \{g(x) + h(a - x)\}.$$

If g and h are lsc and $\operatorname{dom} g \cap \operatorname{dom} h \neq \emptyset$, then by [13], we have that

$$(g \Box h)^* = g^* + h^*, \quad (g + h)^* = \operatorname{cl}(g^* \Box h^*), \tag{2.5}$$

and

$$\operatorname{epi} g^* + \operatorname{epi} h^* \subseteq \operatorname{epi}(g^* \Box h^*) \subseteq \operatorname{cl}(\operatorname{epi} g^* + \operatorname{epi} h^*). \tag{2.6}$$

We end this section with a lemma, which is known in [13] and [15].

Lemma 2.1. *Let $g, h : X \to \bar{\mathbb{R}}$ be proper convex functions satisfying $\operatorname{dom} g \cap \operatorname{dom} h \neq \emptyset$.*

(i) *If g, h are lsc,then*

$$\operatorname{epi}(g + h)^* = \operatorname{cl}(\operatorname{epi} g^* + \operatorname{epi} h^*).$$

(ii) *If either g or h is continuous at some point of $\operatorname{dom} g \cap \operatorname{dom} h$, then*

$$\operatorname{epi}(g + h)^* = \operatorname{epi} g^* + \operatorname{epi} h^*.$$

3 New regularity condition for the Stable Lagrange Duality

Throughout this section, let X, Y be locally convex spaces and $C \subseteq X$ be a nonempty convex set. Let $S \subseteq Y$ be a closed convex cone. Its dual cone $S^\oplus$ is defined by

$$S^\oplus := \{y^* \in Y^* : \langle y^*, y\rangle \ge 0 \text{ for each } y \in S\}.$$

Define an order on Y by saying that $y \le_S x$ if $y - x \in -S$. We attach a greatest element ∞ with respect to $\le_S$ and denote $Y^\bullet := Y \cup \{+\infty\}$. The following operations are defined on $Y^\bullet$: for any $y \in Y$, $y + \infty = \infty + y = \infty$ and $t\infty = \infty$ for any $t \ge 0$. Let $f, g : X \to \bar{\mathbb{R}}$ be proper convex functions such that cl g and $f - g$ are proper, and $h : X \to Y^\bullet$ be S-convex in the sense that for every $u, v \in \operatorname{dom} h$ and every $t \in [0, 1]$,

$$h(tu + (1 - t)v) \le_s th(u) + (1 - t)h(v),$$

(see [16]). Let $\lambda \in S^\oplus$ and let $\operatorname{dom} h := \{x \in X : h(x) \in Y\} \neq \emptyset$. As in [15], we define for each $\lambda \in S^\oplus$,

$$(\lambda h)(x) := \begin{cases} \langle \lambda, h(x)\rangle, & \text{if } x \in \operatorname{dom} h, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that h is S-convex if and only if $(\lambda h)(\cdot) : X \to \bar{\mathbb{R}}$ is a convex function for each $\lambda \in S^{\oplus}$. Following [10], we define the function $h^{\diamond} : X^* \to \bar{\mathbb{R}}$ by

$$h^{\diamond}(x^*) = \inf_{\lambda \in S^{\oplus}} (\lambda h)^*(x^*) \quad \text{for each } x^* \in X^*.$$

Let $h^{-1}(-S) := \{x \in \operatorname{dom} g : h(x) \in -S\}$ and let A denote the solution set of the system $\{x \in C : h(x) \in -S\}$, that is, $A := \{x \in C : h(x) \in -S\}$. To avoid trivially, we always assume that $A \neq \emptyset$. Recall that f is said to be star lsc if λf is lsc on X for each $\lambda \in S^{\oplus}$.

The following lemma, which is taken from [10], will be useful in our study.

Lemma 3.1. *Suppose that h is a proper star lsc and S-convex mapping with $h^{-1}(S) \neq \emptyset$. Then*
(i) *$h^{\diamond}$ is a proper convex function on X^*.*
(ii) *$\operatorname{epi} h^{\diamond}$ is a convex cone.*
(iii) *$\operatorname{epi} \delta^*_{h^{-1}(-S)} = \operatorname{cl}(\operatorname{epi} h^{\diamond})$ and $\operatorname{epi} \delta^*_A = \operatorname{cl}(\operatorname{epi} \delta^*_C + \operatorname{epi} h^{\diamond})$.*

Let $p \in X^*$. Consider the primal problem

$$(P_p) \qquad \begin{array}{ll} \text{Min} & f(x) - g(x) - \langle p, x\rangle, \\ \text{s.t.} & x \in C,\ h(x) \in -S. \end{array} \tag{3.1}$$

Define the Lagrange dual problem by

$$(D_p^L) \quad \inf_{u^* \in \operatorname{dom} g^*} \sup_{\lambda \in S^{\oplus}} \{g^*(u^*) - (f + \delta_C + \lambda h)^*(p + u^*)\}, \tag{3.2}$$

and its Fenchel-Lagrange dual problem by

$$(D_p^{FL}) \quad \inf_{u^* \in \operatorname{dom} g^*} \sup_{\lambda \in S^{\oplus}} \{g^*(u^*) - f^*(x_1^*) - \delta_C^*(x_2^*) - (\lambda h)^*(p + u^* - x_1^* - x_2^*)\}. \tag{3.3}$$

Let $v(P_p)$, $v(D_p^L)$ and $v(D_p^{FL})$ denote the optimal values of problems (P_p), (D_p^L) and (D_p^{FL}), respectively. Let $r \in \mathbb{R}$. By the definition of conjugate function, we have that

$$(p, r) \in \operatorname{epi}(f - g + \delta_A)^* \Leftrightarrow v(P_p) \geq -r. \tag{3.4}$$

Moreover, if g is lsc, then for each $x \in X$,

$$g(x) = g^{**}(x) = \sup_{x^* \in \operatorname{dom} g^*} \{\langle x^*, x\rangle - g^*(x^*)\}. \tag{3.5}$$

Thus, it is easy to see that the following inequalities hold:

$$v(D_p^{FL}) \leq v(D_p^L) \leq v(P_p) \quad \text{for each } p \in X^*, \tag{3.6}$$

that is, the stable weak Lagrange duality and stable weak Fenchel-Lagrange duality hold. However, (3.6) does not necessarily hold in general as showed by [1, Example 3.2]. The following lemma is taken from [1, Lemma 3.3] which gives a sufficient condition to ensure the stable weak Lagrange duality and stable weak Fenchel-Lagrange duality.

Lemma 3.2. *Suppose that the following condition holds:*

$$\operatorname{epi}(f - g + \delta_A)^* = \operatorname{epi}(f - \operatorname{cl} g + \delta_A)^*. \tag{3.7}$$

Then (3.6) holds.

This section is devoted to the study of the stable Lagrange duality and stable Fenchel-Lagrange duality between (P) and (D), which is defined as follows.

Definition 3.1. *It is said that*

(a) *the stable Lagrange duality holds between* (P) *and* (D) *if for each* $p \in X^*$, $v(P_p) = v(D_p^L)$.

(b) *the stable Fenchel-Lagrange duality holds between* (P) *and* (D) *if for each* $p \in X^*$, $v(P_p) = v(D_p^{FL})$.

Below we definite a new constraint qualification to characterize the stable Lagrange duality.

Definition 3.2. *It is said that the family* $\{f, g, \delta_C, h\}$ *satisfies the weaker constraint qualification* ((WCQ) *in brief) if*

$$\operatorname{epi}(f - g + \delta_A)^* \subseteq \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}((f + \delta_C)^* \Box h^\diamond) - (u^*, g^*(u^*))). \tag{3.8}$$

The following proposition provides an equivalent condition to ensure (WCQ) holds.

Proposition 3.1. *Suppose that (3.7) holds* $(e.g., g \text{ is lsc})$ *and that*

$$f \; is \; lsc, h \; is \; star \; lsc, C \; is \; closed. \tag{3.9}$$

Then the family $\{f, g, \delta_C, h\}$ *satisfies* (WCQ) *if and only if*

$$\operatorname{epi}(f - g + \delta_A)^* = \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}((f + \delta_C)^* \Box h^\diamond) - (u^*, g^*(u^*))). \tag{3.10}$$

Proof. To show the equivalence of (WCQ) and (3.10), it suffices to show

$$\bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}((f + \delta_C)^* \Box h^\diamond) - (u^*, g^*(u^*))) \subseteq \operatorname{epi}(f - g + \delta_A)^*. \tag{3.11}$$

To do this, by (3.9), (2.6) and Lemma 2.1(i), it is easy to see that

$$\begin{aligned} \operatorname{epi}((f + \delta_C)^* \Box h^\diamond) &\subseteq \operatorname{cl}(\operatorname{epi}(f + \delta_C)^* + \operatorname{epi} h^\diamond) \\ &\subseteq \operatorname{cl}(\operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \delta_C^*) + \operatorname{epi} h^\diamond) \\ &= \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \delta_C^* + \operatorname{epi} h^\diamond); \end{aligned}$$

while, by Lemma 3.1(iii), one has that

$$\operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \delta_C^* + \operatorname{epi} h^\diamond) = \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} \delta_A^*) = \operatorname{epi}(f + \delta_A)^*,$$

where the second equality holds by Lemma 2.1(i). Thus, by (2.4) and (3.7),

$$\begin{aligned} \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}((f + \delta_C)^* \Box h^\diamond) - (u^*, g^*(u^*))) &\subseteq \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}(f + \delta_A)^* - (u^*, g^*(u^*))) \\ &= \operatorname{epi}(f - \operatorname{cl} g + \delta_A)^* \\ &= \operatorname{epi}(f - g + \delta_A)^*. \end{aligned}$$

Hence, (3.11) holds and the proof is complete. □

Remark 3.1. *To study the Lagrange duality and the Fenchel-Lagrange duality, the author in [1] introduced the following condition* (CQ)*:*

$$\operatorname{epi}(f - g + \delta_A)^* = \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond) - (u^*, g^*(u^*))). \tag{3.12}$$

The following proposition shows that (WCQ) is weaker than (CQ).

Proposition 3.2. *Suppose that f is lsc and C is closed. Then*

$$(CQ) \Rightarrow (WCQ)$$

Proof. Suppose (CQ) holds. By (2.5), we have that

$$(f+\delta_C)^* = \mathrm{cl}(f^*\Box\delta_C^*) \leq f^*\Box\delta_C^*.$$

and

$$(f+\delta_C)^*\Box h^\diamond \leq f^*\Box\delta_C^*\Box h^\diamond.$$

Combing this with (2.3), we see that

$$\mathrm{epi}(f^*\Box\delta_C^*\Box h^\diamond) \subseteq \mathrm{epi}((f+\delta_C)^*\Box h^\diamond).$$

This together with (CQ) implies that (WCQ) holds and the proof is complete. □

The following example shows that (WCQ) is strictly weaker than (CQ).

Example 3.1. *Let $X = Y := \mathbb{R}$ and $C = S := [0,+\infty)$. Let $f, g, h : \mathbb{R} \to \bar{\mathbb{R}}$ be defined by $g := 0$, $h(x) := x^2$ and*

$$f(x) := \begin{cases} +\infty, & x > 0, \\ 2, & x = 0, \\ 0, & x < 0. \end{cases}$$

Then $A := \{x \in C : h(x) \in -S\} = \{0\}$ and for each $x^* \in \mathbb{R}$,

$$f^*(x^*) = \begin{cases} 0, & x^* \geq 0, \\ +\infty, & x^* < 0, \end{cases} \quad \textit{and} \quad (\lambda h)^*(x^*) = \begin{cases} \frac{(x^*)^2}{4\lambda}, & \lambda > 0, \\ \delta_{\{0\}}(x^*), & \lambda = 0. \end{cases}$$

Hence, $h^\diamond = 0$. Note that $(f+\delta_C)^* = (f+\delta_A)^* = -2$. It follows that

$$\mathrm{epi}(f+\delta_A)^* = \mathrm{epi}((f+\delta_C)^*\Box h^\diamond) = \mathbb{R}\times[-2,+\infty).$$

This implies that (WCQ) holds. However, (CQ) does not hold because

$$\mathrm{epi}(f^*\Box\delta_C^*\Box h^\diamond) = \mathbb{R}\times[0,+\infty) \neq \mathrm{epi}(f+\delta_A)^*.$$

The following theorem shows that the condition (CQ) is equivalent to the stable Fenchel-Lagrange duality.

Theorem 3.1. *Suppose that (3.7) and* (3.9) *hold. Then the stable Fenchel-Lagrange duality holds between (P) and (D) if and only if the family $\{f, g, \delta_C, h\}$ satisfies (CQ).*

Proof. Suppose that the stable Fenchel-Lagrange duality holds. Let $(p,r) \in \mathrm{epi}(f-g+\delta_A)^*$. Then, by (3.4), $v(P_p) \geq -r$ and $v(D_p^{FL}) \geq -r$, that is,

$$\inf_{u^*\in\mathrm{dom}\,g^*} \sup_{\lambda\in S^\oplus, x_1^*, x_2^*\in X^*} \{g^*(u^*) - f^*(x_1^*) - \delta_C^*(x_2^*) - (\lambda h)^*(p+u^*-x_1^*-x_2^*)\} \geq -r,$$

This implies that for each $u^* \in \mathrm{dom}\,g^*$,

$$\sup_{\lambda\in S^\oplus, x_1^*, x_2^*\in X^*} \{g^*(u^*) - f^*(x_1^*) - \delta_C^*(x_2^*) - (\lambda h)^*(p+u^*-x_1^*-x_2^*)\} \geq -r.$$

Let $u^* \in \mathrm{dom}\,g^*$ and let $\varepsilon > 0$. Then there exist $\lambda \in S^\oplus$ and $x_1^*, x_2^* \in X^*$ such that

$$g^*(u^*) + r + \varepsilon \geq f^*(x_1^*) + \delta_C^*(x_2^*) + (\lambda h)^*(p+u^*-x_1^*-x_2^*);$$

while, by the definition of the infimal convolution function, we have that

$$\begin{aligned} f^*(x_1^*) + \delta_C^*(x_2^*) + (\lambda h)^*(p + u^* - x_1^* - x_2^*) &\geq (f^* \Box \delta_C^* \Box (\lambda h)^*)(p + u^*) \\ &\geq (f^* \Box \delta_C^* \Box h^\diamond)(p + u^*). \end{aligned}$$

Hence,

$$g^*(u^*) + r + \varepsilon \geq (f^* \Box \delta_C^* \Box h^\diamond)(p + u^*).$$

Letting $\varepsilon \to 0$. we have that

$$(f^* \Box \delta_C^* \Box h^\diamond)(p + u^*) \leq g^*(u^*) + r, \tag{3.13}$$

which implies that

$$(p + u^*, g^*(u^*) + r) \in \operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond),$$

that is,

$$(p, r) \in (\operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond) - (u^*, g^*(u^*))).$$

By the arbitrary of u^*, we can get that

$$(p, r) \in \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond) - (u^*, g^*(u^*))).$$

Thus

$$\operatorname{epi}(f - g + \delta_A)^* \subseteq \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond) - (u^*, g^*(u^*))).$$

Therefore, by [1, Proposition 3.7] (CQ) holds.

Conversely, suppose that the family $\{f, g, \delta_C, h\}$ satisfies (CQ). Let $p \in X^*$. If $v(P_p) = -\infty$, then the stable Fenchel-Lagrange duality holds between (P) and (D) trivially. Below we assume that $r := v(P_p) \in \mathbb{R}$. Then by (3.4), $(p, -r) \in \operatorname{epi}(f - g + \delta_A)^*$ and

$$(p, -r) \in \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond) - (u^*, g^*(u^*))),$$

thanks to (CQ). Let $u^* \in \operatorname{dom} g^*$. Then

$$(p + u^*, g^*(u^*) - r) \in \operatorname{epi}(f^* \Box \delta_C^* \Box h^\diamond),$$

and hence

$$\inf_{x_1^*, x_2^* \in X^*} \{f^*(x_1^*) + \delta_C^*(x_2^*) + h^\diamond(p + u^* - x_1^* - x_2^*)\} \leq g^*(u^*) - r.$$

Take $\varepsilon > 0$. Then there exist $x_1^*,\ x_2^* \in X^*$ such that

$$f^*(x_1^*) + \delta_C^*(x_2^*) + h^\diamond(p + u^* - x_1^* - x_2^*) \leq g^*(u^*) - r + \frac{\varepsilon}{2}; \tag{3.14}$$

while, by the definition of the function $h^\diamond$, there exists $\lambda \in S^\oplus$ such that

$$(\lambda h)^*(p + u^* - x_1^* - x_2^*) \leq h^\diamond(p + u^* - x_1^* - x_2^*) + \frac{\varepsilon}{2}.$$

This together with (3.14) implies that

$$f^*(x_1^*) + \delta_C^*(x_2^*) + (\lambda h)^*(p + u^* - x_1^* - x_2^*) \leq g^*(u^*) - r + \varepsilon,$$

that is,

$$r - \varepsilon \leq g^*(u^*) - f^*(x_1^*) - \delta_C^*(x_2^*) - (\lambda h)^*(p + u^* - x_1^* - x_2^*).$$

Therefore,

$$r - \varepsilon \leq \sup_{\lambda \in S^\oplus x_1^* \in X^* x_2^* \in X^*} \{g^*(u^*) - f^*(x_1^*) - \delta_C^*(x_2^*) - (\lambda h)^*(p + u^* - x_1^* - x_2^*)\}.$$

and

$$r - \varepsilon \leq \inf_{u^* \in \operatorname{dom} g^*} \sup_{\lambda \in S^{\oplus} x_1^* \in X^* x_2^* \in X^*} \{g^*(u^*) - f^*(x_1^*) - \delta_C^*(x_2^*) - (\lambda h)^*(p + u^* - x_1^* - x_2^*)\},$$

since $u^* \in \operatorname{dom} g^*$ is arbitrary. Consequently, by the definition of $v(D_p^{FL})$, one has that $v(D_p^{FL}) \geq r - \varepsilon$ and $v(D_p^{FL}) \geq r$ by the arbitrary of ε. Moreover, by Lemma 3.2, $v(D_p^{FL}) \leq v(P_p)$. Thus, $v(P_p) = v(D_p^{FL})$ and so the stable Fenchel-Lagrange duality holds since $p \in X^*$ is arbitrary. The proof is complete. □

Theorem 3.2. *Suppose that (3.7) holds. Consider the following statements.*
(i) *The family $\{f, g, \delta_C, h\}$ satisfies (WCQ).*
(ii) *The stable Lagrange duality holds between (P) and (D).*
Then (i) $\Rightarrow$ (ii). *Furthermore,* (i) $\Leftrightarrow$ (ii) *if the following condition holds:*

$$\operatorname{epi}(f + \delta_C + \lambda h)^* = \operatorname{epi}(f + \delta_C)^* + \operatorname{epi}(\lambda h)^* \quad \textit{for each } \lambda \in S^{\oplus}. \tag{3.15}$$

Proof. Suppose that (i) holds. Let $p \in X^*$. If $v(P_p) = -\infty$, then the stable Lagrange duality holds between (P) and (D) trivially. Below we assume that $r := v(P_p) \in \mathbb{R}$. Then by (3.4), we have that $(p, -r) \in \operatorname{epi}(f - g + \delta_A)^*$ and that

$$(p, -r) \in \bigcap_{u^* \in \operatorname{dom} g^*} (\operatorname{epi}((f + \delta_C)^* \Box h^{\diamond}) - (u^*, g^*(u^*))),$$

thanks to (WCQ). Let $u^* \in \operatorname{dom} g^*$ be arbitrary. Then

$$(p + u^*, g^*(u^*) - r) \in \operatorname{epi}((f + \delta_C)^* \Box h^{\diamond}).$$

This implies that

$$\inf_{x^* \in X^*} \{(f + \delta_C)^*(x^*) + h^{\diamond}(p + u^* - x^*)\} \leq g^*(u^*) - r.$$

Hence, for each $\varepsilon > 0$, there exists $x^* \in X^*$ such that

$$(f + \delta_C)^*(x^*) + h^{\diamond}(p + u^* - x^*) \leq g^*(u^*) - r + \frac{\varepsilon}{2}; \tag{3.16}$$

while, by the definition of $h^{\diamond}$, one sees that

$$(\lambda h)^*(p + u^* - x^*) \leq h^{\diamond}(p + u^* - x^*) + \frac{\varepsilon}{2}.$$

Thus,

$$(f + \delta_C)^*(x^*) + (\lambda h)^*(p + u^* - x^*) \leq g^*(u^*) - r + \varepsilon.$$

This together with the Young-Fenchel inequality (2.1) implies that for each $x \in X$,

$$\begin{aligned} r - \varepsilon &\leq g^*(u^*) - (f + \delta_C)^*(x^*) - (\lambda h)^*(p + u^* - x^*) \\ &\leq g^*(u^*) - \langle x, x^* \rangle + f(x) + \delta_C(x) - \langle p + u^* - x^*, x \rangle + (\lambda h)(x) \\ &= g^*(u^*) - \langle p + u^*, x \rangle + (f + \delta_C + \lambda h)(x). \end{aligned}$$

Note that the above inequalities and the equality hold for each $x \in X$. It follows that

$$\begin{aligned} r - \varepsilon &\leq g^*(u^*) - \sup_{x \in X} \{\langle p + u^*, x \rangle - (f + \delta_C + \lambda h)(x)\} \\ &= g^*(u^*) - (f + \delta_C + \lambda h)^*(p + u^*). \end{aligned}$$

Thus, by the definition of $v(D_p^{FL})$, we have that $v(D_p^{FL}) \geq r - \varepsilon$ and $v(D_p^{FL}) \geq r$ by the arbitrary of ε. Hence, $v(P_p) = v(D_p^L)$ since $v(D_p^L) \leq v(P_p)$ by Lemma 3.2. Therefore, by the arbitrary of $p \in X^*$, (ii) holds.

Suppose that (3.15) holds. Below we show (ii)⇒(i). To do this, assume that (ii) holds. Let $(p, r) \in \mathrm{epi}(f - g + \delta_A)^*$. By (3.4), we see that $v(P_p) \geq -r$ and $v(D_p^L) \geq -r$. Let $u^* \in \mathrm{dom}\, g^*$. Then

$$\sup_{\lambda \in S^{\oplus}} \{g^*(u^*) - (f + \delta_C + \lambda h)^*(p + u^*)\} \geq -r.$$

Thus, for each $\varepsilon > 0$, there exists $\lambda \in S^{\oplus}$ such that

$$g^*(u^*) - (f + \delta_C + \lambda h)^*(p + u^*) \geq -r - \varepsilon.$$

Let $\varepsilon > 0$. Then

$$(p + u^*, r + g^*(u^*) + \varepsilon) \in \mathrm{epi}(f + \delta_C + \lambda h)^* = \mathrm{epi}(f + \delta_C)^* + \mathrm{epi}(\lambda h)^*, \tag{3.17}$$

where the equality holds by (3.15). Note by (2.6) that

$$\mathrm{epi}(f + \delta_C)^* + \mathrm{epi}(\lambda h)^* \subseteq \mathrm{epi}((f + \delta_C)^* \Box (\lambda h)^*) \subseteq \mathrm{epi}((f + \delta_C)^* \Box h^{\diamond}).$$

If follows from (3.17) that

$$(p + u^*, r + g^*(u^*) + \varepsilon) \in \mathrm{epi}((f + \delta_C)^* \Box h^{\diamond}).$$

Thus,

$$((f + \delta_C)^* \Box h^{\diamond})(p + u^*) \leq r + g^*(u^*) + \varepsilon.$$

Letting $\varepsilon \to 0$. We get $((f + \delta_C)^* \Box h^{\diamond})(p + u^*) \leq r + g^*(u^*)$ and so $(p, r) \in (\mathrm{epi}((f + \delta_C)^* \Box h^{\diamond}) - (u^*, g^*(u^*)))$. Therefore,

$$(p, r) \in \bigcap_{u^* \in \mathrm{dom}\, g^*} (\mathrm{epi}(f + \delta_C)^* \Box h^{\diamond} - (u^*, g^*(u^*))).$$

since $u^* \in \mathrm{dom}\, g^*$ is arbitrary. Which implies that (WCQ) holds. The proof is complete. □

Remark 3.2. *Recall that the author in [1, Theorem 3.12] established the stable Lagrange duality under the assumptions that the condition (CQ) holds and one of the following conditions holds:*

(a) $\mathrm{cont} f \cap A \neq \emptyset$ *and* $\mathrm{cont} h \cap A \neq \emptyset$,

(b) $\mathrm{cont} h \cap A \cap \mathrm{int} C \neq \emptyset$,

where $\mathrm{cont} f$ *(resp.* $\mathrm{cont} h$*) denotes the set of all points at which* f *(resp.* h*) is continuous. Note by Lemma 2.1(ii) that the condition (a) or (b) implies that* (3.15) *holds; while, by Proposition 3.2,* (WCQ) *is weaker than* (CQ)*. Thus, our Theorem 3.2 improves the corresponding result in [1, Theorem 3.12].*

4 Conclusions

In this paper, we introduce a new regularity condition which is weaker than the condition in [1] and show that this new condition is equivalent to the stable Lagrange duality for the primal problem and its dual problem. Moreover, we obtain a sufficient and necessary condition which guarantees the Fenchel-Lagrange duality.

Acknowledgment

The authors are grateful to anonymous reviewers for valuable suggestions and remarks which helped to improve the quality of the paper. The first author was supported in part by the Scientific Research Fund of Hunan Provincial Education Department (grant 13B095) and the Innovative Programmes Foundation of China Jishou University (grant JGY201311). The second author was supported from the National Natural Science Foundation of China (grant 11101186) and the Scientific Research Fund of Hunan Provincial Education Department (grant 13B095).

Competing Interests

Authors have declared that no competing interests exist.

Authors' contributions

This work was carried out in collaboration between the two authors. Author MW collected the literatures and wrote the first draft of the manuscript. Author DF revised and improved the draft of the manuscript. All authors read and approved the manuscript.

References

[1] D. H. Fang. Stable Zero Lagrange Duality for DC Conic Programming. Journal of Applied Mathematics. doi:10.1155/2012/606457.

[2] N. Dinh, B. S. Mordukhovich, T. T. A. Nghia. Qualification and optimality conditions for DC programs with infinite constraints. Acta Mathematica Vietnamica. 2009;1(34):125-155.

[3] N. Dinh, B. S. Mordukhovich, T. T. A. Nghia. Subdifferentials of value functions and optimality conditions for DC and bilevel infinite and semi-infinite programs. Mathematical Programming. 2010;(123):101-138.

[4] N. Dinh, T. T. A. Nghia, G. Vallet. A closedness condition and its applications to DC programs with convex constraints. Optimization. 2010;3-4(59):541-560.

[5] N. Dinh, G. Vallet, T. T. A. Nghia. Farkas-type results and duality for DC programs with convex constraints. Journal of Convex Analysis. 2008;2(15):235-262.

[6] R. I. Boţ, G. Wanka. An alternative formulation for a new closed cone constraint qualification. Nonlinear Analysis. 2006;6(64):1367-1381.

[7] R. I. Boţ, S. -M. Grad, G. Wanka. On strong and total Lagrange duality for convex optimization problems. Journal of Mathematical Analysis and Applications. 2008;2(337):1315-1325.

[8] D. H. Fang, C. Li, K. F. Ng. Constraint qualifications for optimality conditions and total Lagrange dualities in convex infinite programming. Nonlinear Analysis. 2010;5(73):1143-1159.

[9] P. Tseng. Some convex programs without a duality gap. Mathematical Programming. 2009;(116):553-578.

[10] V. Jeyakumar, G. Y. Li. New dual constraint qualifications characterizing zero duality gaps of convex programs and semidefinite programs. Nonlinear Analysis. 2009;12(71):2239-2249.

[11] V. Jeyakumar, H. Wolkowicz. Zero duality gaps in infinite-dimensional programming. Journal of Optimization Theory and Applications. 1990;1(67):87-108.

[12] V. Jeyakumar, G. Y. Li. Stable zero duality gaps in convex programming:complete dual characterisations with applications to semidefinite programs. Journal of Mathmatical Analysis and Application. 2009;1(360):156-167.

[13] C. Zâlinescu. Convex Analysis in General Vector Spaces. World Scientific, River Edge, NJ, USA. 2002.

[14] D. H. Fang, X. Q. Yang. Stable and total Fenchel duality for DC optimization problems in locally convex spaces. SIAM Journal on optimization. 2011;3(21):730-760.

[15] D. H. Fang, C. Li, K. F. Ng. Constraint qualifications for extended Farkas's lemmas and Lagrangian dualities in convex infinite programming. SIAM Journal on Optimization. 2009;3(20):1311-1332.

[16] V. Jeyakumar, G. M. Lee, N. Dinh. New sequential Lagrange multiplier conditions characterizing optimality without constraint qualification for convex programs. SIAM Journal on optimization. 2003;2(14):534-547.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=448&id=6&aid=3817