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New Approaches for Solving Interpolation Problems and Homogeneous Linear Recurrence Relations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This article presents a new approach to address the resolution of homogeneous linear recurrences of higher order and interpolation problems. By establishing an explicit formula for the entries of the inverse of generalized Vandermonde matrices, a fresh perspective on these mathematical challenges is introduced. The study primarily focuses on linear recurrence relations and thoroughly investigates cases involving characteristic polynomials with both simple roots and roots of multiplicity. To illustrate the effectiveness and practicality of the proposed method, a comprehensive set of illustrative examples is provided, highlighting its applicability in solving a wide range of instances of linear recurrence relations. Additionally, the limitations of the formula are discussed, particularly in scenarios where its applicability may be restricted. The findings of this study contribute significantly to the existing literature, providing an alternative and promising approach for solving problems that rely on the inverse Vandermonde matrix. In conclusion, this article emphasizes the need for further research to explore the computational advantages of the proposed method and to extend its applicability to cases featuring characteristic polynomials with a single root of multiplicity greater than one.

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By expanding the knowledge in the field, this study offers valuable insights into the resolution of linear recurrences and interpolation problems, presenting a new perspective and expanding the existing knowledge in the field.

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1 Introduction

There are many mathematical problems that calculation depends on to solve a system of linear equations. Although Babylon used the systems in daily life, the systems of linear equations that we know today arose in Europe in 1637 by Renê Descartes [1]. He introduced Cartesian Geometry where the lines and planes are represented by linear equations and computing their intersections amounts to solving a system.

The progress in the studies of systems of linear equations came from determinants. Well-known mathematicians such as Leibniz and Lagrange in 17^{th} century participated in this development. However, more than 50 years after Leibniz the results that we use in Linear Algebra were presented by Cramer when he showed how to solve an $n \times n$ system based on determinants [2].

In order for matrix algebra to develop Cayley in 1855 defined matrix multiplication and published the idea of the identity matrix as well as the inverse of a square matrix, [see more about the History of Systems of Linear Equation in [3]].

There are several methods for inverting matrices. Most of all methods, in which the solution is a result of a finite number of arithmetic operations, can be classified as methods of factorization and methods of modification, [4]. The process of inverting a matrix usually is not an easy task, so in order to improve the performance of this task, some special matrices are individually studied. Our interest is in the Vandermonde matrix. In special cases, the explicit formula for the entries of inverse of the Vandermonde matrix can be provided, in a classic way, in terms of determinants, [5].

We call Vandermonde matrix $V = (V_{i,j})$ the matrix of order $n \times n$ in the form

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix}, \tag{1.1}$$

or $V_{i,j} = \alpha_i^{j-1} \in \mathbb{C}$, for all $1 \leq i, j \leq n$. This well-known matrix is important because the elements V_{ij} depends on the variables $\alpha_1, \dots, \alpha_n$, which forms a geometric progression.

The Vandermonde matrix appears in many different circumstances as polynomial interpolation, least square regression, construction of error-detecting and error-correcting codes, and solving systems of differential equations with constant coefficients, [see [6] and references therein].

The name of this matrix is devoted to Alexandre Theóphile Vandermonde (1735-1796). His entire mathematical contribution consisted of four published articles. Especially in the fourth paper, of title Mémorie sur l'elimination, Vandermonde discusses a general method for solving linear equation systems using alternating functions, but the Vandermonde matrix is not considered in any of Vandermonde's works, [see [6] and references therein].

In fact, the determinant of the Vandermonde matrix, called the Vandermonde determinant, is given by

$$\left|V\right| = \prod_{1\leq i<j\leq n} (x_j - x_i), \tag{1.2}$$

thus $|V| \neq 0,$ which implies that every system involving a Vandermonde matrix has a solution and it is unique. There are several proofs for Expression (1.2). In a simple way, the formula can be proved by induction on the order of matrix $n,$ [7]. In the case $n = 2,$ is verified

$$\left|V\right| = v_{1,1}v_{2,2} - v_{1,2}v_{2,1} = x_2 - x_1 = \prod_{1\leq i<j\leq 2} (x_j - x_i).$$

A process for inverting the generalized Vandermonde matrix related to a specific linear difference equation was established in [8]. In [9] the authors discussed the inversion method presented in [8] from the point of view of solving some usual difference equations. In addition, the method of inverting a generalized Vandermonde matrix, using the analytic properties of a fundamental system related to specific linear difference equations, was provided in [10].

In this context, we establish a study on the resolution of homogeneous linear recurrences of higher order and interpolation problems via the new method published for inverting the Vandermonde matrix associated with the Fibonacci fundamental system. Our goal is to discuss the explicit formulas for entries of inverted generalized Vandermonde matrix presented in [10] and provide a new approach for solving a linear recurrence relation and interpolation problems, which depends on the process of inverting Vandermonde matrices. For linear recurrence relation, the three cases are discussed: the polynomial characteristic associated with only simple roots, the polynomial characteristic associated with two or more roots where at least one of them has multiplicity greater than one, and the polynomial characteristic associated with one single root with multiplicity greater than one. We finish establishing that the formula for the case with one single root with multiplicity greater than one is not applied and the solution is determined in the traditional way.

The content of this paper is organized as follows. In Section 2 the method of inverting the Vandermonde matrix is presented and the explicit formulas of entries of the matrix are established for when the roots of the characteristic polynomial associated are simple. Section 3 is devoted to studying the application of the method to solve an interpolation problem. Section 4 presents the application of the method to solve linear recurrence relations. Examples concerned with simple roots and roots with multiplicity greater than one are considered.

2 The Inverse of Generalized Vandermonde Matrices

2.1 On invertible matrices

It is well-known the definition of an invertible matrix. Recalling, let A a square matrix of order $n,$ (or $n \times n,$ n lines and n columns) it is defined as the inverse matrix associated to $A,$ the matrix $B,$ of order n such that $AB = BA = Id_n$ where Id_n is the identity matrix of order $n.$ If B exist, we say that A is an invertible matrix.

There are several algorithms to establish the inverse matrix, such as Gaussian elimination, Gauss-Jordan methods, LU decomposition, and Cholesky decomposition as shown in [11, 12]. The most efficient methods consist of expressing matrix A as a product of two factors P and Q such as $PA = Q,$ where Q is easily inverted.

In mathematical terms, if we can describe a problem as an equation to be solved, namely, $AX = H,$ where X and H are matrices of order $n \times 1,$ just find out if A has an inverse. If there is an inverse matrix $B,$ of $A,$ then the solution of the problem is unique and given by $X = BH,$ solving the problem. In fact, in the study of problems associated with matrices with a large order $n \geq 100$, the researcher worries about the type of matrix and the optimum algorithm for each case. Then, new approaches have great acceptance in the literature, even if some of them are not great for all types and orders of matrices. For our focus, in the next subsection, we will discuss the approaches to find the inverse of the Vandermonde matrix.

2.2 On inverse of Vandermonde matrices

There are various types of generalized Vandemonde matrices proposed and studied in the literature. Here we are considering a generalized Vandermonde matrix V given by the following form

$$V = \begin{bmatrix} 1 & 0 & ... & 0 & ... & 1 & 0 & ... & 0 \\ \lambda_1 & \lambda_1 & ... & \lambda_1 & ... & \lambda_s & \lambda_s & ... & \lambda_s \\ \lambda_1^2 & 2\lambda_1^2 & ... & 2^{m_1-1}\lambda_1^2 & ... & \lambda_s^2 & 2\lambda_s^2 & ... & 2^{m_s-1}\lambda_s^2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \lambda_1^{r-1} & (r-1)\lambda_1^{r-1} & ... & (r-1)^{m_1-1}\lambda_1^{r-1} & ... & \lambda_s^{r-1} & (r-1)\lambda_s^{r-1} & ... & (r-1)^{m_s-1}\lambda_s^{r-1} \end{bmatrix}, \tag{2.1}$$

where the entries $V_{i,j} \in \mathbb{R}$, for all $1 \leq i, j \leq n$.

This matrix can be associated with a recursive problem with linear coefficients where its characteristic polynomial is $p(\lambda) = \prod_{i=1}^{s}(\lambda - \lambda_i)^{m_i} = \lambda^r - \sum_{i=0}^{r-1} a_i \cdot \lambda^i$. This relationship provided a new perspective for the approaches involving the Vandermonde matrix using the results in the theory of linear difference equations. The first approach was shown in [8], where the explicit formulas for the entries of the inverse of Vandermonde matrix were provided depending on the roots of characteristic polynomial associated. The method was applied for some special cases in [9] and a new point of view for solving linear recurrence relations was established.

Since the given explicit formulas were derived from results of the generalized Fibonacci sequences, in paper [10] was established a process for inverting the generalized Vandermonde matrix, using the analytic properties of a fundamental system, the bases of vectorial space of the generalized Fibonacci sequences. The principal results are resumed in the theorem below,

[Proposition 2.5 and Proposition 2.6, [10]]

Let V be a generalized Vandermonde matrix as in equation (2.1), then its inverse is given by,

$$V^{-1} = \begin{bmatrix} \beta_{1,0}^{(0)} & \beta_{1,0}^{(1)} & ... & \beta_{1,0}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{1,m_1-1}^{(0)} & \beta_{1,m_1-1}^{(1)} & ... & \beta_{1,m_1-1}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{s,0}^{(0)} & \beta_{s,0}^{(1)} & ... & \beta_{s,0}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{s,m_s-1}^{(0)} & \beta_{s,m_s-1}^{(1)} & ... & \beta_{s,m_s-1}^{(r-1)} \end{bmatrix}, \tag{2.2}$$

with each entry of the matrix is

$$\beta_{i,k}^{(r-1)} = \sum_{t=k}^{m_i-1} s(t,k) \cdot \frac{\gamma_t^{[i]}(\lambda_1, ..., \lambda_s)}{t! \cdot \lambda_i^t}, \tag{2.3}$$

with $s(t,k)$ being the Stirling number of the first kind, namely, the number of permutations on t elements with k cycles,

$$s(t,k) = \begin{bmatrix} n \\ k \end{bmatrix} \tag{2.4}$$

and $\gamma_k^{[i]}$ is defined by

$$\gamma_k^{[i]}(\lambda_1, ..., \lambda_s) = (-1)^{r-m_i} \cdot \sum_{\varepsilon_k^{[i]}} \left(\prod_{i \le j \ne i \le s} \frac{\binom{n_j + m_j - 1}{n_j}}{(\lambda_j - \lambda_i)^{n_j + m_j}} \right), \tag{2.5}$$

where $\varepsilon_k^{[i]} = \{(n_1, ..., n_s) \in \mathbb{N}^{s-1}; n_1 + ... + n_{i-1} + n_{i+1} + ... + n_s = m_i - k - 1\}$, and

$$\beta_{i,j}^{(p)} = \sum_{k=0}^{p} a_{r-p-1+k} \cdot C_{i,j}^{(k+1)}, \tag{2.6}$$

where

$$C_{i,j}^{(d)} = \lambda_i^{-d} \cdot \sum_{k=j}^{m_i - 1} (-1)^{k-j} \beta_{i,k}^{(r-1)} \binom{k}{j} \cdot d^{k-j}, \tag{2.7}$$

and a_i is the coefficients in the characteristic polynomial $p(\lambda) = \lambda^r - \sum_{i=0}^{r-1} a_i \cdot \lambda^i$.

Since many known problems of recursive sequences like Fibonacci sequence or polynomial interpolation problems might result in a simple Vandermonde matrix, where all the roots of the characteristic polynomial are simple roots, it is possible to reduce the expressions given in Theorem 2.2.

Let V be a Vandermonde matrix given by

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \dots & \lambda_m^{n-1} \end{bmatrix}, \tag{2.8}$$

with $\lambda_i \ne \lambda_j$, for all $i \ne j$, $1 \le i, j \le n$. Then its inverse is given by,

$$V^{-1} = \begin{bmatrix} \beta_{1,0}^{(0)} & \beta_{1,0}^{(1)} & \dots & \beta_{1,0}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{1,m_1-1}^{(0)} & \beta_{1,m_1-1}^{(1)} & \dots & \beta_{1,m_1-1}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{s,0}^{(0)} & \beta_{s,0}^{(1)} & \dots & \beta_{s,0}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{s,m_s-1}^{(0)} & \beta_{s,m_s-1}^{(1)} & \dots & \beta_{s,m_s-1}^{(r-1)} \end{bmatrix}, \tag{2.9}$$

with each enter of the matrix given as follows,

$$\beta_{i,0}^{(r-1)} = \frac{(-1)^{r-1}}{\prod_{0 \le j \ne i \le s} (\lambda_j - \lambda_i)}, \tag{2.10}$$

$$\beta_{i,0}^{(p)} = \sum_{k=0}^{p} a_{r-p-1+k} \cdot \frac{\beta_{i,0}^{(r-1)}}{\lambda_i^{k+1}}. \tag{2.11}$$

Proof. In fact, consider a characteristic polynomial with r roots, all with multiplicity one, $m_i = 1$, with $1 \leq i \leq r$, then Equation (2.3) is simplified as follows,

$$\beta_{i,0}^{(r-1)} = s(0,0) \cdot \frac{\gamma_0^{[1]}(\lambda_1, ..., \lambda_r)}{0! \cdot \lambda_i^0} \Rightarrow \beta_{i,0}^{(r-1)} = \gamma_0^{[i]}(\lambda_1, ..., \lambda_r).$$

Notice the set $\varepsilon_0^{[i]}$ is given by the solution of $n_1 + ... + n_{i-1} + n_{i+1} + ... + n_r = 0$, with n_k being a non negative integer for $1 \leq k \leq r$, implying $\varepsilon_0^{[i]} = \{(0, ..., 0)\}$. Then, using Equation (2.5),

$$\gamma_0^{[i]}(\lambda_1, ..., \lambda_r) = (-1)^{r-1} \cdot \left[\frac{\binom{n_1}{n_1}}{(\lambda_1 - \lambda_i)} \cdot \frac{\binom{n_2}{n_2}}{(\lambda_2 - \lambda_i)} \cdot ... \cdot \frac{\binom{n_r}{n_r}}{(\lambda_r - \lambda_i)} \right]$$

$$\Rightarrow \gamma_0^{[i]}(\lambda_1, ..., \lambda_r) = \frac{(-1)^{r-1}}{(\lambda_1 - \lambda_i) \cdot ...(\lambda_{i-1} - \lambda_i) \cdot (\lambda_{i+1} - \lambda_i) \cdot ... \cdot (\lambda_r - \lambda_i)}.$$

Following with Equation (2.6) and using the previous results, it is obtained,

$$\beta_{i,0}^{(p)} = a_{r-p-1} \cdot C_{i,0}^{(1)} + ... + a_{r-1} \cdot C_{i,0}^{(p+1)},$$

with coefficients $C_{i,0}^{(d)}$ obtained by Equation (2.7),

$$C_{i,0}^{(d)} = \lambda_i^{-d} \cdot \beta_{i,0}^{(r-1)} \cdot d^0 \Rightarrow C_{i,0}^{(d)} = \lambda^{-d} \cdot \beta_{i,0}^{(r-1)}.$$

The previous results lead to the following expression,

$$\beta_{i,0}^{(p)} = a_{r-p-1} \cdot \frac{\beta_{i,0}^{(r-1)}}{\lambda_i} + ... + a_{r-1} \cdot \frac{\beta_{i,0}^{(r-1)}}{\lambda_i^{p+1}}.$$

Thus, the entries of the inverse of the Vandermonde matrix are given as shown in equations (2.10) e (2.11). □

The Proposition 2.2 show us explicit formulas for inverting a Vandemonde matrix in the case of the recurrence relation associated has a characteristic polynomial with simple roots. The following sections are devoted to applying the results of Theorem 2.2 and Proposition 2.2 in order to obtain a new way to discuss an interpolation problem and recurrence relation. Observe that in Expression (2.5) and Expression (2.10) is not explicit that can be applied when there is a single root with multiplicity m_i.

3 An Interpolation Problem

Consider the given ordered pairs (x_i, y_i) with $1 \leq i \leq n$, then the interpolation polynomial problem consists in finding a polynomial $P(x) = \sum_{i=0}^{n-1} a_i x^i$ with at most degree $n - 1$ such that $P(x_i) = y_i$. In other words, the problem results in solving the system

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}, \tag{3.1}$$

where the variables a_i are the coefficients of $P(x) = \sum_{i=0}^{n-1} a_i x^i$.

Then the interpolation polynomial problem results in determining the inverse Vandermonde matrix. This problem has only one solution. An important case in problems of polynomial interpolation is when the first n positive integers are known. In this case, the matrix associated with the problem has the following form,

$$V^t = \begin{bmatrix} 1 & 1 & ... & 1 \\ 1 & 2 & ... & 2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & n & ... & n^{n-1} \end{bmatrix}.$$

Notice the transpose of V^t is a Vandermone matrix, so it is possible to determine the solution of this interpolation problem by studying the inverse of matrix V. The polynomial associated with V is

$$p(\lambda) = (\lambda - 1) \cdot ... \cdot (\lambda - n) = \lambda^n - \sum_{k=0}^{n-1} a_k \lambda^{n-k-1},$$

obtaining coefficients $a_k = (-1)^k \cdot s(n+1, r-k)$, where $s(n,k)$ is the Stirling number of first kind.

In this situation, equation (2.10) can be rewritten as presented in the sequence,

$$\beta_{k,0}^{(n-1)} = \frac{(-1)^{n-k}}{(k-1)! \cdot (n-k)!}.$$

Then, for an interpolation problem where the first n positive integers terms of the sequence are known, equation (2.10) is simplified as shown in follows equation,

$$\beta_{i,0}^{(n-1)} = \frac{(-1)^{n-i}}{(i-1)! \cdot (n-i)!}. \tag{3.2}$$

As a general case of order 3, consider the interpolation problem where $p(1) = a_1^2 = \alpha_1, p(2) = a_1^2 + a_2^2 = \alpha_2, p(3) = a_1^2 + a_2^2 + a_3^2 = \alpha_3$, and $p(4) = a_1^2 + a_2^2 + a_3^2 + a_4^2 = \alpha_4$, it results in the following linear system,

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{b}.$$

Notice that the matrix A is the transpose matrix of a Vandermonde matrix V, which means this problem can be solved by determining V^{-1} and evaluating $x = (V^{-1})^t \cdot b$. By definition of a Vandermonde matrix is obtained $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, and $\lambda_4 = 4$. Then the polynomial p related to matrix V is such that $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$, which gives the coefficients $a_0 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, a_1 = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4), a_2 = \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$, and $a_3 = -(\lambda_1\lambda_2\lambda_3\lambda_4)$, which comes from Girard's relation.

Now, through Equation (3.2),

$$\beta_{1,0}^{(3)} = \frac{(-1)^3}{(-1)^0 \cdot 0! \cdot 3!} \Rightarrow \beta_{1,0}^{(3)} = -\frac{1}{6},$$

$$\beta_{2,0}^{(3)} = \frac{(-1)^3}{(-1)^1 \cdot 1! \cdot 2!} \Rightarrow \beta_{1,0}^{(3)} = \frac{1}{2},$$

$$\beta_{3,0}^{(3)} = \frac{(-1)^3}{(-1)^2 \cdot 2! \cdot 1!} \Rightarrow \beta_{1,0}^{(3)} = -\frac{1}{2},$$

$$\beta_{4,0}^{(3)} = \frac{(-1)^3}{(-1)^3 \cdot 3! \cdot 0!} \Rightarrow \beta_{1,0}^{(3)} = \frac{1}{6},$$

By Equation (2.11), coefficients $\beta_{i,0}^{(k)}$ are obtained, for $i = 1$,

$$\beta_{1,0}^{(0)} = a_3 \cdot \frac{\beta_{1,0}^{(3)}}{\lambda_1^1} \Rightarrow \beta_{1,0}^{(0)} =,$$

$$\beta_{1,0}^{(1)} = a_2 \cdot \frac{\beta_{1,0}^{(3)}}{\lambda_1^1} + a_3 \cdot \frac{\beta_{1,0}^{(3)}}{\lambda_1^2} \Rightarrow \beta_{1,0}^{(1)} = -\frac{13}{3},$$

$$\beta_{1,0}^{(2)} = a_1 \cdot \frac{\beta_{1,0}^{(3)}}{\lambda_1^1} + a_2 \cdot \frac{\beta_{1,0}^{(3)}}{\lambda_1^2} + a_3 \cdot \frac{\beta_{1,0}^{(3)}}{\lambda_1^3} \Rightarrow \beta_{1,0}^{(2)} = \frac{3}{2},$$

for $i = 2$,

$$\beta_{2,0}^{(0)} = a_3 \cdot \frac{\beta_{2,0}^{(3)}}{\lambda_2^1} \Rightarrow \beta_{2,0}^{(0)} = -6,$$

$$\beta_{2,0}^{(1)} = a_2 \cdot \frac{\beta_{2,0}^{(3)}}{\lambda_2^1} + a_3 \cdot \frac{\beta_{2,0}^{(3)}}{\lambda_2^2} \Rightarrow \beta_{2,0}^{(1)} = \frac{19}{2},$$

$$\beta_{2,0}^{(2)} = a_1 \cdot \frac{\beta_{2,0}^{(3)}}{\lambda_2^1} + a_2 \cdot \frac{\beta_{2,0}^{(3)}}{\lambda_2^2} + a_3 \cdot \frac{\beta_{2,0}^{(3)}}{\lambda_2^3} \Rightarrow \beta_{2,0}^{(2)} = -4,$$

for $i = 3$,

$$\beta_{3,0}^{(0)} = a_3 \cdot \frac{\beta_{3,0}^{(3)}}{\lambda_3^1} \Rightarrow \beta_{3,0}^{(0)} = 4,$$

$$\beta_{3,0}^{(1)} = a_2 \cdot \frac{\beta_{3,0}^{(3)}}{\lambda_3^1} + a_3 \cdot \frac{\beta_{3,0}^{(3)}}{\lambda_3^2} \Rightarrow \beta_{3,0}^{(1)} = -7,$$

$$\beta_{3,0}^{(2)} = a_1 \cdot \frac{\beta_{3,0}^{(3)}}{\lambda_3^1} + a_2 \cdot \frac{\beta_{3,0}^{(3)}}{\lambda_3^2} + a_3 \cdot \frac{\beta_{3,0}^{(3)}}{\lambda_3^3} \Rightarrow \beta_{3,0}^{(2)} = \frac{7}{2},$$

for $i = 4$,

$$\beta_{4,0}^{(0)} = a_3 \cdot \frac{\beta_{4,0}^{(3)}}{\lambda_4^1} \Rightarrow \beta_{4,0}^{(0)} = -1,$$

$$\beta_{4,0}^{(1)} = a_2 \cdot \frac{\beta_{4,0}^{(3)}}{\lambda_4^1} + a_3 \cdot \frac{\beta_{4,0}^{(3)}}{\lambda_4^2} \Rightarrow \beta_{4,0}^{(1)} = \frac{11}{6},$$

$$\beta_{4,0}^{(2)} = a_1 \cdot \frac{\beta_{4,0}^{(3)}}{\lambda_4^1} + a_2 \cdot \frac{\beta_{4,0}^{(3)}}{\lambda_4^2} + a_3 \cdot \frac{\beta_{4,0}^{(3)}}{\lambda_4^3} \Rightarrow \beta_{4,0}^{(2)} = -1.$$

Since the coefficients of V^{-1} were determined, the interpolation polynomial is determined as presented by,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -6 & \frac{19}{2} & -4 & \frac{1}{2} \\ 4 & -7 & \frac{7}{2} & -\frac{1}{2} \\ -1 & \frac{11}{6} & -1 & \frac{1}{6} \end{bmatrix}^t \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4\ \alpha_1 - 6\ \alpha_2 + 4\ \alpha_3 - \alpha_4 \\ \frac{1}{6}(-26\ \alpha_1 + 57\ \alpha_2 - 42\ \alpha_3 + 11\ \alpha_4) \\ \frac{1}{2}(3\ \alpha_1 - 8\ \alpha_2 + 7\ \alpha_3 - 2\ \alpha_4) \\ \frac{1}{6}(-\alpha_1 + 3\ \alpha_2 - 3\ \alpha_3 + \alpha_4) \end{bmatrix}.$$

Returning to the original variables, the coefficients are $a = a_1^2 - 3\ a_2^2 + 3\ a_3^2 - a_4^2$, $b = \frac{13}{3}\ a_2^2 - \frac{31}{6}\ a_3^2 + \frac{11}{6}\ a_4^2$, $c = -\frac{3}{2}\ a_2^2 + \frac{5}{2}\ a_3^2 - a_4^2$, and $d = \frac{1}{6}\ a_2^2 - \frac{1}{3}\ a_3^2 + \frac{1}{6}\ a_4^2$.

Therefore, the desired polynomial p is given by

$$p(x) = a_1^2 - 3a_2^2 + 3a_3^2 - a_4^2 + \frac{(26a_2^2 - 31a_3^2 + 11a_4^2)}{6}x + \frac{(-3a_2^2 + 5a_3^2 - 2a_4^2)}{2}x^2 + \frac{(a_2^2 - 2a_3^2 + a_4^2)}{6}x^3.$$

The following example consists of a particular case of obtaining an explicit formula for the sum of an arithmetic sequence of a high order, which can be seen as a case of an interpolation problem.

Example 3.1. *Consider the problem where is desired to obtain an explicit formula for the sum of the first n squared terms of an arithmetic sequence $(a_n)_{n\in\mathbb{N}}$. Since $(a_n)_{n\in\mathbb{N}}$ is an arithmetic sequence, follows that $a_n = a_1 + (n-1)r$, where a_1 is the first term of this sequence and r is the common difference of successive members.*

Then the problem consists in obtaining an explicit formula for S_n, given by,

$$S_n = \sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} (a_1^2 + 2 \cdot a_1(k-1)r + (k-1)^2 \cdot r^2). \tag{3.3}$$

Observe that S_n is an arithmetic sequence of the third order, which means it can be described as a third-degree polynomial $p(x) = a + bx + cx^2 + dx^3$, since

$$\Delta S_k = S_k - S_{k-1} \Rightarrow \Delta S_k = a_k^2$$
$$\Delta^2 S_k = a_k^2 - a_{k-1}^2 \Rightarrow \Delta^2 S_k = (4r^2 - 2a_1 r)k - r^2$$
$$\Delta^3 S_k = 2a_1 r - 4r^2,$$

which r is constant.

Notice the problem of the sum of squares of the first n positive integers, is a particular problem of this case, where $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_4 = 4$. In this particular situation is obtained $a = 0$, $b = \frac{1}{6}$, $c = \frac{1}{2}$, and $d = \frac{1}{3}$. Then, the polynomial to describe the summation is given by,

$$p(n) = 0 + \frac{1}{6}\ x + \frac{1}{2}\ x^2 + \frac{1}{3}\ x^3 \Rightarrow p(n) = \frac{n(n+1)(2n+1)}{6},$$

as it is mostly known.

It seems to us that this approach to solve interpolation problems and its application to solve recurrence problems are new.

4 Solving Linear Recurrence Relations

This section is devoted to solving a linear recurrence relation using the method of obtaining the explicit formulas for the entries of the inverse Vandermonde matrix described in Section 2. Consider a linear recurrence relation of order r

$$V_n = a_0 V_{n-1} + a_{n-2} V_{n-} + \cdots + a_{r-1} V_{n-r+1}, \tag{4.1}$$

for $n \geq r$, where the coefficients a_i, $i = 0, 1, \cdots n-1$ are real or complex numbers, $a_{r-1} \neq 0$, and $V_0, V_1, V_2, \cdots, V_{r-1}$ are the initial conditions.

The characteristic polynomial associated with (4.1) is given by

$$z_n - a_0 z_{n-1} - a_{n-2} z_{n-} + \cdots - a_{r-1} z_{n-r+1} = 0. \tag{4.2}$$

It is well-known that the explicit analytic formula for V_n can be derived by a linear combination of the $n-$th power roots of the characteristic polynomial associated (Binet formula) with the constants of the combination given as a solution of a Vandermonde system. In the sequel, each possible case of the roots of characteristic polynomial, simple, and roots with multiplicity, are discussed and the explicit analytic formulas are provided.

4.1 Simple roots

Recall the linear recurrence relation of order r given by the Equation (4.1) and initial conditions $V_0, V_1, V_2, \cdots, V_{r-1}$. Suppose that the characteristic polynomial (4.2) associated have simple roots $\lambda_1, \lambda_2, \cdots \lambda_r$. Then, this implies the solution given by

$$V_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n + \cdots C_r \cdot \lambda_r^n.$$

where the constants C_i are derived as the solution of the system

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & x_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix}.$$

By direct application of Proposition 2.2 we obtain,

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} \beta_{1,0}^{(0)} & \beta_{1,0}^{(1)} & \dots & \beta_{1,0}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{1,m_1-1}^{(0)} & \beta_{1,m_1-1}^{(1)} & \dots & \beta_{1,m_1-1}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{s,0}^{(0)} & \beta_{s,0}^{(1)} & \dots & \beta_{s,0}^{(r-1)} \\ \vdots & \vdots & & \vdots \\ \beta_{s,m_s-1}^{(0)} & \beta_{s,m_s-1}^{(1)} & \dots & \beta_{s,m_s-1}^{(r-1)} \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix} \tag{4.3}$$

with each enter of the matrix defined by Equations (2.10) and (2.11).

The linear recurrence relation of the Fibonacci numbers is considered a special application.

Example 4.1. *Consider the following recursive problem known as the Fibonacci sequence, where its terms are generated by adding the two immediately previous terms, resulting in the recursive equation* $F_n = F_{n-1} + F_{n-2}$, *and* $F_0 = F_1 = 1$.

The characteristic polynomial of the given recursive problem is given by $p(z) = z^2 - z - 1$, *in which the coefficients are* $a_0 = 1$, $a_1 = 1$, *and its roots* $\lambda_1 = \frac{1+\sqrt{5}}{2}$ *and* $\lambda_2 = \frac{1-\sqrt{5}}{2}$. *This implies the solution of this recursive sequence is* $F_n = C_1 \cdot \lambda_1^n + C_2 \cdot \lambda_2^n$, *where* C_1 *and* C_2 *are constants to be determined.*

Since the roots of the characteristic polynomial are all simple, it is possible to use Equation (2.10) *to solve this problem,*

$$\beta_{1,0}^{(1)} = \frac{(-1)^{2-1}}{-\sqrt{5}} \Rightarrow \beta_{1,0}^{(1)} = \frac{1}{\sqrt{5}},$$

$$\beta_{2,0}^{(1)} = \frac{(-1)^{2-1}}{\sqrt{5}} \Rightarrow \beta_{2,0}^{(1)} = -\frac{1}{\sqrt{5}}.$$

Obtained the coefficients $\beta_{i,0}^{(1)}$ *, the missing coefficients are determined by Equation* (2.11)*,*

$$\beta_{1,0}^{(0)} = a_1 \cdot \frac{\beta_{1,0}^{(1)}}{\lambda_1} \Rightarrow \beta_{1,0}^{(0)} = 1 \cdot \frac{\left(\frac{1}{\sqrt{5}}\right)}{\left(\frac{1+\sqrt{5}}{2}\right)} \Rightarrow \beta_{1,0}^{(0)} = \frac{2}{\sqrt{5}+5}$$

$$\beta_{2,0}^{(0)} = a_1 \cdot \frac{\beta_{2,0}^{(1)}}{\lambda_2} \Rightarrow \beta_{2,0}^{(0)} = 1 \cdot \frac{\left(\frac{-1}{\sqrt{5}}\right)}{\left(\frac{1-\sqrt{5}}{2}\right)} \Rightarrow \beta_{2,0}^{(0)} = \frac{-2}{\sqrt{5}-5}.$$

The constants C_1 *and* C_2 *of this problem are given by the product between the inverse of the associated Vandermonde matrix and the initial condition of the sequence as follows,*

$$\begin{bmatrix} \beta_{1,0}^{(0)} & \beta_{1,0}^{(1)} \\ \beta_{2,0}^{(0)} & \beta_{2,0}^{(1)} \end{bmatrix} \cdot \begin{bmatrix} F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}+5} & \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}-5} & -\frac{1}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right) \\ -\frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix}.$$

Thus,

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right).$$

4.2 Roots with multiplicity greater than one: two or more different roots

Given the linear recurrence relation of order r given by the Equation (4.1) and initial conditions $V_0, V_1, V_2, \cdots, V_{r-1}$, Suppose that the characteristic polynomial (4.2) associated have roots $\lambda_1, \lambda_2, \cdots \lambda_s$ with multiplicities $m_1, m_2, \cdots, m_s$, respectivally, where $m_1 + m_2 + \cdots + m_s = n$. Then, this implies the solution is given under the form,

$$V_n = (C_{1,1} + C_{1,2}n + \cdots + C_{1,m_1}n^{m_1-1})\lambda_1^n + \cdots + (C_{s,1} + C_{s,2}n + \cdots + C_{s,m_s}n^{m_s-1})\lambda_s^n$$

where the constants $C_{i,j}$ is derived as solution of the system

$$\begin{bmatrix} 1 & 0 & ... & 0 & ... & 1 & ... & 0 \\ \lambda_1 & \lambda_1 & ... & \lambda_1 & ... & \lambda_s & ... & \lambda_s \\ \lambda_1^2 & 2\lambda_1^2 & ... & 2^{m_1-1}\lambda_1^2 & ... & \lambda_s^2 & ... & 2^{m_s-1}\lambda_s^2 \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \lambda_1^{r-1} & (r-1)\lambda_1^{r-1} & ... & (r-1)^{m_1-1}\lambda_1^{r-1} & ... & \lambda_s^{r-1} & ... & (r-1)^{m_s-1}\lambda_s^{r-1} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_{n-1} \end{bmatrix}.$$

By direct application of Proposition 2.2 to solve (4.2), we derive the explicity formula for V_n.

Let the following illustrative example clarify the approach.

Example 4.2. *Consider the problem of determining the solution of a recursive sequence described as* $v_n = 4v_{n-1} + 3v_{n-2} - 18v_{n-3}$, *with* $v_0 = 0$ *and* $v_1 = v_2 = 1$.

The characteristic polynomial associated with this recursive sequence is $p(\lambda) = \lambda^3 - 4\lambda^2 - 3\lambda + 18$, *obtaining* $a_0 = 4$, $a_1 = 3$, $a_2 = -18$, $\lambda_1 = -2$, $\lambda_2 = 3$, $m_1 = 1, m_2 = 2$, *and the solution of the problem* $v_n = C_1 \cdot (-2)^n + C_2 \cdot 3^n + C_3 \cdot n3^n$, *where* C_1, C_2, C_3 *are the constants to be determined by the presented general method.*

By Equation (2.3), *the coefficients* $\beta_{i,j}^{(2)}$ *are obtained as follows, for* $i = 1$,

$$\beta_{1,0}^{(2)} = s(0,0) \cdot \frac{\gamma_0^{[1]}(\lambda_1, \lambda_2)}{0! \cdot \lambda_1^0} \Rightarrow \beta_{1,0}^{(2)} = \gamma_0^{[1]}(\lambda_1, \lambda_2),$$

for $i = 2$ *and* $j = 0$,

$$\beta_{2,0}^{(2)} = s(0,0) \cdot \frac{\gamma_0^{[2]}(\lambda_1, \lambda_2)}{0! \cdot \lambda_2^0} + s(1,0) \cdot \frac{\gamma_1^{[2]}(\lambda_1, \lambda_2)}{1! \cdot \lambda_2^1} \Rightarrow \beta_{2,0}^{(2)} = \gamma_0^{[2]}(\lambda_1, \lambda_2),$$

for $i = 2$ *and* $j = 1$,

$$\beta_{2,1}^{(2)} = s(1,1) \cdot \frac{\gamma_1^{[2]}(\lambda_1, \lambda_2)}{1! \cdot \lambda_2^1} \Rightarrow \beta_{2,0}^{(2)} = \frac{\gamma_1^{[2]}(\lambda_1, \lambda_2)}{\lambda_2}.$$

The set $\varepsilon_0^{[1]}, \varepsilon_0^{[2]}$, *and* $\varepsilon_1^{[2]}$ *used in Equation* (2.5) *are defined as* $\{n_2 = 0\}, \{n_1 = 1\}$, *and* $\{n_1 = 0\}$, *respectively.*

Now, by Equation (2.5), *the coefficients previously obtained are rewritten as follows,*

$$\beta_{1,0}^{(2)} = (-1)^{3-1} \cdot \frac{\binom{n_2+m_2-1}{n_2}}{(\lambda_2 - \lambda_1)^{n_2+m_2}} \Rightarrow \beta_{1,0}^{(2)} = \frac{1}{(\lambda_2 - \lambda_1)^2} \Rightarrow \beta_{1,0}^{(2)} = \frac{1}{25},$$

$$\beta_{2,0}^{(2)} = (-1)^{3-2} \cdot \frac{\binom{n_1+m_1-1}{n_1}}{(\lambda_1 - \lambda_2)^{n_1+m_1}} \Rightarrow \beta_{2,0}^{(2)} = -\frac{1}{(\lambda_1 - \lambda_2)^2} \Rightarrow \beta_{2,0}^{(2)} = -\frac{1}{25},$$

$$\beta_{2,1}^{(2)} = (-1)^{3-2} \cdot \frac{\binom{n_1+m_1-1}{n_1}}{(\lambda_1 - \lambda_2)^{n_1+m_1}} \cdot \frac{1}{\lambda_2} \Rightarrow \beta_{2,1}^{(2)} = -\frac{1}{(\lambda_1 - \lambda_2) \cdot \lambda_2} \Rightarrow \beta_{1,0}^{(2)} = \frac{1}{15}.$$

Obtained the coefficients $\beta_{i,j}^{(2)}$, *by Formula* (2.6), *the other coefficients are given by* $\beta_{i,j}^{(0)} = a_2 \cdot C_{i,j}^{(1)}$ *and* $\beta_{i,j}^{(1)} = a_1 \cdot C_{i,j}^{(1)} + a_2 \cdot C_{i,j}^{(2)}$. *Then, by applying Equation* (2.7), *the coefficientes* $C_{i,j}^{(d)}$ *are determined and, consequently, so does* $\beta_{i,j}^{k}$.

For $i = 1$, *we have,*

$$C_{1,0}^{(1)} = \lambda_1^{-1} \cdot (-1)^{0-0} \beta_{1,0}^{(2)} \binom{0}{0} \cdot 1^{0-0} \Rightarrow C_{1,0}^{(1)} = -\frac{1}{50},$$

$$C_{1,0}^{(2)} = \lambda_1^{-2} \cdot (-1)^{0-0} \beta_{1,0}^{(2)} \binom{0}{0} \cdot 2^{0-0} \Rightarrow C_{1,0}^{(2)} = \frac{1}{100},$$

$$\beta_{1,0}^{(0)} = a_2 \cdot C_{1,0}^{(1)} \Rightarrow \beta_{1,0}^{(0)} = -18 \cdot \left(-\frac{1}{50}\right) \Rightarrow \beta_{1,0}^{(0)} = \frac{9}{25},$$

$$\beta_{1,0}^{(1)} = a_1 \cdot C_{1,0}^{(1)} + a_2 \cdot C_{1,0}^{(2)} \Rightarrow \beta_{1,0}^{(1)} = 3 \cdot \left(-\frac{1}{50}\right) - 18 \cdot \left(\frac{1}{100}\right) \Rightarrow \beta_{1,0}^{(1)} = -\frac{6}{25}.$$

For $i = 2$ *and* $j = 0$,

$$C_{2,0}^{(1)} = \lambda_2^{-1} \cdot \left((-1)^{0-0} \beta_{2,0}^{(2)} \binom{0}{0} \cdot 1^{0-0} + (-1)^{1-0} \beta_{2,1}^{(2)} \binom{1}{0} \cdot 1^{1-0} \right) \Rightarrow C_{2,0}^{(1)} = -\frac{8}{225},$$

$$C_{2,0}^{(2)} = \lambda_2^{-2} \cdot \left((-1)^{0-0} \beta_{2,0}^{(2)} \binom{0}{0} \cdot 2^{0-0} + (-1)^{1-0} \beta_{2,1}^{(2)} \binom{1}{0} \cdot 2^{1-0} \right) \Rightarrow C_{2,0}^{(2)} = -\frac{13}{675},$$

$$\beta_{2,0}^{(0)} = a_2 \cdot C_{2,0}^{(1)} \Rightarrow \beta_{2,0}^{(0)} = -18 \cdot \left(-\frac{8}{225}\right) \Rightarrow \beta_{2,0}^{(0)} = \frac{16}{25}$$

$$\beta_{2,0}^{(1)} = a_1 \cdot C_{2,0}^{(1)} + a_2 \cdot C_{2,0}^{(2)} \Rightarrow \beta_{2,0}^{(1)} = 3 \cdot \left(-\frac{8}{225}\right) - 18 \cdot \left(-\frac{-13}{675}\right) \Rightarrow \beta_{2,0}^{(1)} = \frac{6}{25},$$

and for $i = 2$ *and* $j = 1$, *we obtain*

$$C_{2,1}^{(1)} = \lambda_2^{-1} \cdot (-1)^{1-1} \beta_{2,1}^{(2)} \binom{1}{1} \cdot 1^{1-1} \Rightarrow C_{2,1}^{(1)} = \frac{1}{45},$$

$$C_{2,1}^{(2)} = \lambda_2^{-2} \cdot (-1)^{1-1} \beta_{2,1}^{(2)} \binom{1}{1} \cdot 2^{1-1} \Rightarrow C_{2,1}^{(2)} = \frac{1}{135},$$

$$\beta_{2,1}^{(0)} = a_2 \cdot C_{2,1}^{(1)} \Rightarrow \beta_{2,1}^{(0)} = -18 \cdot \left(\frac{1}{45}\right) \Rightarrow \beta_{2,1}^{(0)} = -\frac{2}{5},$$

$$\beta_{2,1}^{(1)} = a_1 \cdot C_{2,1}^{(1)} + a_2 \cdot C_{2,1}^{(2)} \Rightarrow \beta_{2,1}^{(1)} = 3 \cdot \frac{1}{45} - 18 \cdot \frac{1}{135} \Rightarrow \beta_{2,1}^{(1)} = -\frac{1}{15}.$$

Then, the desired coefficients are obtained by the product between the inverse matrix with coefficients $V^{-1} = (\beta_{i,j}^{(k)})$ *and the initial conditions,*

$$\begin{bmatrix} C_1 \\ C2 \\ C3 \end{bmatrix} = \begin{bmatrix} \frac{9}{25} & -\frac{6}{25} & \frac{1}{25} \\ \frac{16}{25} & \frac{6}{25} & -\frac{1}{25} \\ -\frac{2}{5} & -\frac{1}{15} & \frac{1}{15} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C2 \\ C3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ \frac{1}{5} \\ 0 \end{bmatrix}.$$

Therefore, the recursive sequence is described as $v_n = -\frac{1}{5} \cdot (-2)^n + \frac{1}{5} \cdot (3)^n$.

4.3 Roots with multiplicity greater than one: single root

A special case to take into consideration when working with this new method is shown in the sequence. Let the recursive sequence given by the Equation (4.1) with the characteristic polynomial associated given by $p(z) = (z-\lambda)^r$ and initial conditions $V_0, V_1, \cdots V_{r-1}$. Observe that the root of $p(z)$ is λ with multiplicity r. Therefore, the solution to this problem is associated with the following Vandermonde matrix,

$$\begin{bmatrix} 1 & 0 & ... & 0 \\ \lambda & \lambda & ... & \lambda \\ \lambda^2 & 2\lambda^2 & ... & 2^{r-1}\lambda^2 \\ \vdots & \vdots & & \vdots \\ \lambda^{r-1} & (r-1)\lambda^{r-1} & ... & (r-1)^{r-1-1}\lambda^{r-1} \end{bmatrix}. \tag{4.4}$$

Observe that the determinant of matrix (4.4) is given by $sf(r-1)\lambda^{r(r-1)/2} \neq 0$ where $sf(n) = 1!2!3! \cdots n!$ is a superfactorial of n.

The matrix expressed under the form (4.4) is of type of generalized Vandermonde matrix (2.1), nevertheless, the given formula presented in (2.5) cannot be used in this situation to obtain its inverse since it does not include a case where there is a single solution to the characteristic equation associated with the recursive sequence. It is also possible to verify that using the value one, since it is the identity number of multiplication, in the denominator does not generate the correct answer to the problem.

In this situation, it would be necessary to use one of the classical methods to solve. In this situation, the constants that determine the solution of the recursive problem can be obtained by solving the following linear system.

Example 4.3. *To clarify, consider the following example. Let the recursive sequence with the characteristic polynomial given by* $p(\lambda) = (\lambda - 2)^4$ *and initial conditions* $v_3 = v_2 = v_1 = v_0 = 1$.

The zeros of polynomial p *are given by* $\lambda = 2$ *and its multiplicity* $m = 4$, *therefore, the solution to this problem is associated with the following matrix,*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 2^2 & 2\cdot 2^2 & 2^2\cdot 2^2 & 2^3\cdot 2^2 \\ 2^3 & 3\cdot 2^3 & 3^2\cdot 2^3 & 3^3\cdot 2^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 4 & 8 & 16 & 32 \\ 8 & 24 & 72 & 216 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 4 & 8 & 16 & 32 \\ 8 & 24 & 72 & 216 \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

By the Gauss elimination method,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 4 & 8 & 16 & 32 & 1 \\ 8 & 24 & 72 & 216 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & -1 \\ 0 & 8 & 16 & 32 & -3 \\ 0 & 24 & 72 & 216 & -7 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & -1 \\ 0 & 0 & 8 & 24 & 1 \\ 0 & 0 & 48 & 192 & 5 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & -1 \\ 0 & 0 & 8 & 24 & 1 \\ 0 & 0 & 0 & 48 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ \frac{3}{16} \\ -\frac{1}{48} \end{bmatrix}$$

Concluding, the problem is described by the following function,

$$v_n = \left(1 - \frac{2n}{3} + \frac{3n^2}{16} - \frac{n^3}{48}\right) \cdot 2^n.$$

This example presents a possibility to extend the study done by the main article discussed here in order to include this special case in its formulation.

5 Conclusions

In this study, we presented a new perspective on the solution of problems involving the inverse of Vandermonde matrices. We discussed the explicit formulas for entries of inverted generalized Vandermonde matrix presented in [10] and provided, as applications, a new approach for solving a linear recurrence relation and interpolation problems, which depends on the process of inverting Vandermonde matrices. We described the interpolation problem and established, with a special case, the solution. An example in the mathematical context was exhibited. The cases of each kind of root multiplicity of characteristic polynomial associated with the recurrence relation were discussed and a new approach was established.

It seems to us that this new approach is not currently in the literature, and the results and examples established here can be used as an alternative method to solve the problems that depend on the inverse Vandermonde matrix.

The discussion of if this method has computational advantages over others in the literature is an open problem. It seems to us that the complexity time of the methods presented in [8] and [10] is the same and the LU method is best for problems with the order of the matrix associated is large. In addition, to extend the study done by the main article discussed here in order to include the case of the characteristic polynomial that has a single root with multiplicity greater than one.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Rabouin D. What descartes knew of mathematics in 1628. Historia Mathematica. 2010;37(3):428-459.

[2] Kosinski AA. Cramer's rule is due to cramer. Mathematics Magazine. 2001;74(4):310-312.

[3] Vitulli M. A brief history of linear algebra and matrix theory. Department of Mathematics. University of Oregon; 2012.

[4] Householder AS. A survey of some closed methods for inverting matrices. Journal of the Society for Industrial and Applied Mathematics. 1957;5(3):155–169.
Available: http://www.jstor.org/stable/2098997

[5] Knuth DE. Knuth. The art of computer programming. Fundamental Algorithms. 1997;1.

[6] Lundengard K. Generalized vandermonde matrices and determinants in eletromagnetic compatibility. Mälardalen University Press Licentiate Theses. 2017;253.

[7] Bellman R. Introduction to matrix analysis. McGraw-Hill Book Company, New York, 2nd edition; 1970.

[8] Ben Taher R, Rachidi M. Solving some generalized Vandermonde systems and inverse of their associate matrices via new approaches for the Binet formula. Applied Mathematics and Computation. 2016;290:267-280.

[9] Sa LSC, Spreafico EVP. Sobre um novo método inversão de matrizes de Vandermonde para solução de recorrências lineares homogêneas de ordem superior. Professor de Matemática Online. 2020;8:64-72.

[10] Aniz C, Rachidi M. Inverse of the generalized Vandermonde matrix via the fundamental system of linear difference equations. International Journal of Advanced Engineering Research and Science (IJAERS). 2022;9(6).

[11] Bunch JR, Hopcroft JE. Triangular factorization and inversion by fast matrix multiplication. Mathematics of Computation. 1974;28(125).

[12] Krishnamoorthy A, Menon D. Matrix inversion using Cholesky decomposition. Signal Processing: Algorithms, Architectures, Arrangements, and Applications (SPA). 2013;70-72.

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