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Solving The Hyperbolic Telegraph Equation Using a Modified Adomian Decomposition Method with an Invertible Partial Differential Operator

Saleem Nasser Alomari [a*] and Yahya Qaid Hasan [b]

[a] *Department of Mathematics, Faculty of Education and Sciences, Albaydha University, Yemen.*
[b] *Department of Mathematics, Faculty of Applied Science, Thamar University, Yemen.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, a modified Adomian decomposition method (MADM) for solving the hyperbolic telegraph equation is proposed. The MADM introduces a new inverse partial differential operator that can speed up the convergence rate of the standard ADM. We also present a technique for converting the equation to a special case form, which makes the MADM easier to implement. The proposed method was tested on six different linear and nonlinear telegraph equations in one and two dimensions. The results show that the method is accurate and efficient for solving the telegraph equation.

Keywords: Hyperbolic telegraph equation; modified Adomian decomposition method; convergence; accuracy.

**Corresponding author: E-mail: saleemnasser86@gmail.com;*

1 Introduction

Most physical systems can be mathematically modeled using linear or nonlinear partial differential equations (PDEs) in various scientific fields. Hyperbolic PDEs are among the most important types of PDEs [1]. One example of a hyperbolic PDE is the second-order telegraph equation with constant coefficients. This equation is a powerful tool that can be used to describe a wide range of phenomena in various fields.

For instance, the telegraph equation can model the random motion of a particle in fluid flow, the transmission of electrical impulses in nerve and muscle cells, and the propagation of electromagnetic waves in superconducting media. It can also describe the propagation of pressure waves in pulsatile blood flow in arteries [2]. The study of the telegraph equation is an active area of research. Developing new methods for solving it can lead to significant advances in our understanding of these and other physical phenomena.

Solving the telegraph equation analytically is not always possible or convenient, especially when the boundary and initial conditions are complicated or nonlinear. Therefore, various numerical and approximate methods have been proposed and used to obtain solutions of the telegraph equation [3, 4, 5, 6, 7, 8, 9, 10, 11], such as finite difference methods, Runge-Kutta methods, perturbation methods, homotopy methods, and Adomian decomposition method (ADM). Among these methods, ADM is a popular and powerful technique that can handle linear and nonlinear problems without linearization or discretization. It provides the solution as an infinite series that converges rapidly and has easily computable components [12]. However, ADM also has some limitations and drawbacks, such as the limited choice of acceptable linear operators and initial approximations, the difficulty of integrating higher order deformation equations, and the need of using the so-called Adomian polynomials. To overcome these challenges and improve the accuracy and efficiency of ADM, several modifications have been proposed by various researchers. One of them is the modified Adomian decomposition method (MADM), which was introduced by Hasan and Zhu in 2009 [13].This modification is based on developing a new invertible differential operator and was introduced for solving second-order ordinary differential equations with constant coefficients . The main objective of this paper is to use the differential operator introduced in [14] and use it to apply this modified method for the hyperbolic telegraph equation. This method has some advantages over ADM, such as simplifying the calculation process, giving exact solutions for some equations by using only a few iterations, and with this method only the initial conditions are needed for finding the solution. In this paper, we apply the MADM to solve the hyperbolic telegraph equation. We show that MADM is more efficient than ADM for solving this equation. We also present numerical results that demonstrate the accuracy and efficiency of the MADM.

This paper is organized as follows: in Sect. 2 the analysis of the method is given with two cases being considered. In Sect. 3 a proposed technique for converting the problem from the general case to the special case is introduced. Numerical results are presented and discussed in Sect. 4. and, finally, conclusions are summarized in Sect. 5.

2 Analysis of The Method

In this work, we consider the general hyperbolic telegraph equation of the form

$$\frac{\partial^2 u}{\partial t^2} + a\,\frac{\partial u}{\partial t} + b\,u = c\,\frac{\partial^2 u}{\partial x^2} + f(x,t) + F(u(x,t)), \tag{2.1}$$

with initial conditions as follows:

$$u(x,0) = g_1(x) \quad , \quad u_t(x,0) = g_2(x), \tag{2.2}$$

where a,b and c are constants related to the inductance, capacitance and conductance of the cable respectively [1], f, g_1, g_2 are known functions, and the unknowing function u can be voltage or current through the wire at position x and time t, and F(u(x,t))represents the nonlinear terms [8].

Under the transformation $a = \alpha + \beta$ and $b = \alpha\beta$ Eq.(2.1) becomes

$$\frac{\partial^2 u}{\partial t^2} + (\alpha+\beta)\,\frac{\partial u}{\partial t} + \alpha\beta\,u = c\,\frac{\partial^2 u}{\partial x^2} + f(x,t) + F(u(x,t)) \tag{2.3}$$

we will consider two cases

2.1 First case when ($\alpha \neq \beta$) :

In this general case we propose the new differential operator $L_t(.)$ as follows

$$L_t(.) \ = \ e^{-\alpha t} \ \frac{\partial}{\partial t} \ e^{-(\beta-\alpha)t} \ \frac{\partial}{\partial t} \ e^{\beta t} \ (.). \tag{2.4}$$

Applying this operator to u results in

$$\begin{aligned} L_t \ (u) \ &= \ e^{-\alpha t} \ \frac{\partial}{\partial t} \ e^{-(\beta-\alpha)t} \ \frac{\partial}{\partial t} \ [e^{\beta t}u] \\ &= \ e^{-\alpha t} \ \frac{\partial}{\partial t} \ e^{-(\beta-\alpha)t} \ [\ u_t \ e^{\beta t} \ + \ \beta \ u \ e^{\beta t} \ ] \\ &= \ e^{-\alpha t} \ \frac{\partial}{\partial t} \ [\ u_t \ e^{\alpha t} \ + \ \beta \ u \ e^{\alpha t} \ ] \\ &= \ e^{-\alpha t} \ [\ u_{tt} \ e^{\alpha t} \ + \ \alpha \ u_t \ e^{\alpha t} \ + \ \beta \ u_t \ e^{\alpha t} \ + \ \alpha\beta \ u \ e^{\alpha t} \ ] \\ &= \ u_{tt} \ + \ (\alpha+\beta) \ u_t \ + \ \alpha\beta \ u \ . \end{aligned}$$

So, under this operator the left hand side of Eq.(2.3) becomes $L_t u$ so the telegraph Eq.(2.3) can be written as

$$L_t u = c \ u_{xx} + f(x,t) \ + F(u(x,t)). \tag{2.5}$$

The inverse operator L_t^{-1} is therefore considered a two-fold integral operator, as below,

$$L_t^{-1}(.) = e^{-\beta t} \int_0^t e^{(\beta-\alpha)t} \int_0^t e^{\alpha t}(.) \ dtdt \ . \tag{2.6}$$

Applying L_t^{-1} to the left hand side of the Eq.(2.3)

$$L_t^{-1}(u_{tt} + (\alpha+\beta)u_t + \alpha\beta u) = e^{-\beta t} \int_0^t e^{(\beta-\alpha)t} \int_0^t e^{\alpha t}(u_{tt} + (\alpha+\beta)u_t + \alpha\beta u) \ dtdt$$

$$= e^{-\beta t} \textstyle\int_0^t e^{(\beta-\alpha)t}[e^{\alpha t}u_t + \beta e^{\alpha t}u - u_t(x,0) - \beta u(x,0)] \ dt$$

$$= e^{-\beta t} \textstyle\int_0^t [e^{\beta t}u_t + \beta e^{\beta t}u - e^{(\beta-\alpha)t}u_t(x,0) - \beta e^{(\beta-\alpha)t}u(x,0)] \ dt$$

$$= e^{-\beta t}[e^{\beta t}u - u_(x,0) - \frac{1}{\beta-\alpha}e^{(\beta-\alpha)t}u_t(x,0) - \frac{1}{\beta-\alpha}\beta e^{(\beta-\alpha)t}u(x,0) + \frac{1}{\beta-\alpha}u_t(x,0) + \frac{\beta}{\beta-\alpha}u(x,0)]$$

$$= u - \frac{1}{\beta-\alpha}e^{-\alpha t}u_t(x,0) - \frac{\beta}{\beta-\alpha}e^{-\alpha t}u(x,0) + \frac{1}{\beta-\alpha}e^{-\beta t}u_t(x,0) + \frac{\alpha}{\beta-\alpha}e^{-\beta t}u(x,0) \ .$$

Operating with L_t^{-1} on Eq.(2.5) we get

$$\begin{aligned} u = \frac{1}{\beta-\alpha}e^{-\alpha t}u_t(x,0) + \frac{\beta}{\beta-\alpha}e^{-\alpha t}u(x,0) - \frac{1}{\beta-\alpha}e^{-\beta t}u_t(x,0) - \frac{\alpha}{\beta-\alpha}e^{-\beta t}u(x,0) \\ +cL_t^{-1}(u_{xx}) + L_t^{-1}(f(x,t)) \ + L_t^{-1}(F(u(x,t))). \end{aligned}$$

Substituting the initial conditions (2.2) in the above equation, then

$$\begin{aligned} u \ = \frac{1}{\beta-\alpha} \ e^{-\alpha t} \ g_2(x) + \frac{\beta}{\beta-\alpha} \ e^{-\alpha t} \ g_1(x) - \frac{1}{\beta-\alpha} \ e^{-\beta t} \ g_2(x) - \frac{\alpha}{\beta-\alpha} \ e^{-\beta t} \ g_1(x) \\ + c \ L_t^{-1}(u_{xx}) \ + \ L_t^{-1}(f(x,t)) \ + L_t^{-1}(F(u(x,t))). \end{aligned} \tag{2.7}$$

The ADM suggests that the unknown linear function u may be represented by the decomposition series

$$\sum_{k=0}^{\infty} u_k,$$

where the components $u_k, k \geq 0$ can be computed recursively, and the nonlinear term $F(u(x,t))$ can be expressed by an infinite series of the so-called Adomian polynomials A_k given in the form [12]

$$F(u(x,t)) = \sum_{k=0}^{\infty} A_k(u_0, u_1, u_2, ..., u_k),$$

where the Adomian polynomials A_k can be evaluated by using the following expression

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{k}\lambda^i u_i\right)\right]_{\lambda=0}, \quad k = 0,1,2,.... \tag{2.8}$$

Therefore the solution in a series form is

$$\begin{aligned}\sum_{k=0}^{\infty} u_k \; =& \frac{1}{\beta-\alpha}e^{-\alpha t}g_2(x) + \frac{\beta}{\beta-\alpha}e^{-\alpha t}g_1(x) - \frac{1}{\beta-\alpha}e^{-\beta t}g_2(x) - \frac{\alpha}{\beta-\alpha}e^{-\beta t}g_1(x) \\ &+ L_t^{-1}(f(x,t)) \; + cL_t^{-1}(\sum_{k=0}^{\infty}(u_k)_{xx}) + L_t^{-1}(\sum_{k=0}^{\infty}(A_k).\end{aligned}$$

Through using ADM, the components $u_k(x,t)$ can be determined as

$$u_0 = \frac{1}{\beta-\alpha}\; e^{-\alpha t}\; g_2(x) + \frac{\beta}{\beta-\alpha}\; e^{-\alpha t}\; g_1(x) - \frac{1}{\beta-\alpha}\; e^{-\beta t}\; g_2(x) - \frac{\alpha}{\beta-\alpha}\; e^{-\beta t}\; g_1(x) + \; L_t^{-1}(f(x,t))$$

and

$u_{k+1} \; = c\; L_t^{-1}\; (\; (u_k)_{xx}\;) + L_t^{-1}A_k, \quad k = 0,1,2,3,....$

Once we have determined the components of $u(x,t)$, the solution in a series form is established by summing up these iterations. This series could provide the exact solution in a closed form.

The solution $u(x,t)$ can be approximated by the truncated series:

$$\phi_k = \sum_{m=0}^{k-1} u_m, \quad \lim_{k\to\infty}\phi_k = u(x,t).$$

2.2 Second case when ($\alpha = \beta$) :

In this special case the telegraph equation has the form

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha\frac{\partial u}{\partial t} + \alpha^2 u = c\;\frac{\partial^2 u}{\partial x^2} + f(x,t) + F(u(x,t)), \tag{2.9}$$

with initial conditions as follows:

$$u(x,0) = g_1(x) \quad , \quad u_t(x,0) = g_2(x). \tag{2.10}$$

So the differential operator becomes

$$L_t(.) \;=\; e^{-\alpha t}\;\frac{\partial}{\partial t}\;\frac{\partial}{\partial t}\;e^{\alpha t}\;(.). \tag{2.11}$$

And the inverse operator is

$$L_t^{-1}(.) = e^{-\alpha t}\int_0^t\int_0^t e^{\alpha t}(.)\ dtdt. \tag{2.12}$$

Applying L_t^{-1} of (2.12) to the left-hand side of Eq.(2.9) we find

$$\begin{aligned} L_t^{-1}[\ u_{tt}\ +\ 2\alpha\ u_t\ +\ \alpha^2\ u] & \\ &= e^{-\alpha t}\int_0^t\int_0^t e^{\alpha t}(u_{tt}+2\alpha u_t+\alpha^2 u)\ dtdt \\ &= e^{-\alpha t}\int_0^t (e^{\alpha t}u_t+\alpha e^{\alpha t}u-u_t(x,0)-\alpha u(x,0))\ dt \\ &= u-te^{-\alpha t}u_t(x,0)-e^{-\alpha t}u(x,0)-\alpha te^{-\alpha t}u(x,0). \end{aligned} \tag{2.13}$$

In an operator form Eq.(2.9) is written as

$$L_t(u) = c\ \frac{\partial^2 u}{\partial x^2}+f(x,t)+F(u(x,t)) \tag{2.14}$$

Operating with L_t^{-1} on Eq.(2.14) results in

$$\begin{aligned} u =& te^{-\alpha t}u_t(x,0)+e^{-\alpha t}u(x,0)+\alpha te^{-\alpha t}u(x,0)+L_t^{-1}f(x,t) \\ &+c\ L_t^{-1}u_{xx}+L_t^{-1}F(u(x,t)) \end{aligned} \tag{2.15}$$

Substituting the initial conditions (2.10) in Eq.(2.15) we get

$$\begin{aligned} u =& te^{-\alpha t}\ g_2(x)+e^{-\alpha t}\ g_1(x)+\alpha t\ e^{-\alpha t}\ g_1(x)+L_t^{-1}f(x,t) \\ &+c\ L_t^{-1}u_{xx}+L_t^{-1}F(u(x,t)) \end{aligned} \tag{2.16}$$

By the ADM the solution u is considered as an infinite series $\sum_{k=0}^{\infty}u_k$, so Eq.(2.16) becomes

$$\begin{aligned} \sum_{k=0}^{\infty}u_k =& te^{-\alpha t}\ g_2(x)+e^{-\alpha t}\ g_1(x)+\alpha t\ e^{-\alpha t}\ g_1(x)+L_t^{-1}f(x,t) \\ &+c\ L_t^{-1}(\sum_{k=0}^{\infty}u_k)_{xx}+L_t^{-1}\sum_{k=0}^{\infty}A_k. \end{aligned} \tag{2.17}$$

The components of u is given by

$$\begin{aligned} u_0 &= te^{-\alpha t}\ g_2(x)+e^{-\alpha t}\ g_1(x)+\alpha t\ e^{-\alpha t}\ g_1(x)+L_t^{-1}f(x,t) \\ u_{k+1} &= L_t^{-1}((u_k)_{xx})+L_t^{-1}A_k, \quad k=0,1,2,\ldots. \end{aligned}$$

3 Special Case Transformation Technique

Our research indicates that implementing the inverse partial differential operator 2.12 of the special case is simpler, as it involves only the exponential function raised to the power α. This simplifies the calculation of iterations and can sometimes lead to faster determination of the exact solution.

As such, we propose converting the problem to the special case form before solving it. This can be achieved by rewriting the telegraph equation so that the coefficient of the dependent variable u is equal to the square of half the coefficient of its first derivative u_t, as follows :

Suppose the telegraph equation has the form

$$u_{tt} + au_t + bu = cu_{xx} + f(x,t) + F(u(x,t)). \tag{3.1}$$

First we rewrite the Eq.(3.1) by adding $(\frac{a}{2})^2 u$ to its both sides as follows:

$$u_{tt} + 2 * (\frac{a}{2})u_t + bu + (\frac{a}{2})^2 u = (\frac{a}{2})^2 u + cu_{xx} + f(x,t) + F(u(x,t))$$

$$u_{tt} + 2(\frac{a}{2})u_t + (\frac{a}{2})^2 u = [(\frac{a}{2})^2 - b]u + cu_{xx} + f(x,t) + F(u(x,t)).$$

We put $\frac{a}{2} = \alpha$ and $(\frac{a}{2})^2 - b = \gamma$ then Eq.(3.1) will take the form

$$u_{tt} + 2\alpha\ u_t + \alpha^2\ u = \gamma\ u + c\ u_{xx} + f(x,t) + F(u(x,t)). \tag{3.2}$$

It is clear that Eq.(3.2) is still equivalent to the original Eq.(3.1) [15].

Now the differential operator $L_t(.)$ becomes

$$L_t(.) = e^{-\alpha t}\ \frac{\partial^2}{\partial t^2}\ e^{\alpha t}\ (.). \tag{3.3}$$

The inverse operator L_t^{-1} is therefore considered a two-fold integral operator, as below

$$L_t^{-1}(.) = e^{-\alpha t} \int_0^t \int_0^t e^{\alpha t}(.)\ dtdt, \tag{3.4}$$

and the solution u according to (2.16) is

$$\begin{aligned} u =& te^{-\alpha t} g_2(x) + e^{-\alpha t} g_1(x) + \alpha t e^{-\alpha t} g_1(x) + \gamma L_t^{-1} u \\ &+ cL_t^{-1}(u_{xx}) + L_t^{-1} f(x,t) + L_t^{-1} F(u(x,t)). \end{aligned} \tag{3.5}$$

It is obvious that the iterations made by this technique is easier to calculate because u_0 and L_t^{-1} only contain the exponential function raised to the power α and the solution converges faster as demonstrated by the following numerical examples.

4 Numerical Examples

In this section, we present some numerical examples to illustrate the application of the MADM to solve the telegraph equation in different cases with different initial conditions.We compared the MADM with the exact solutions. We also show the convergence and accuracy of the MADM by computing the absolute error for some examples.

4.1 Example 1:

Consider the following one-dimensional telegraph equation [3]

$$u_{tt} + 2\ u_t + 2\ u = u_{xx} + xe^{-t}, \quad x \in [0,1], \tag{4.1}$$

with initial conditions

$$u(x,0) = x \quad , \quad u_t(x,0) = -x.$$

Here we have $\alpha+\beta=2$ and $\alpha\beta=2$
$\Rightarrow \alpha=1+i$ and $\beta=1-i$. Substituting in (2.6) then the inverse operator is

$$L_t^{-1}(.)=e^{-(1-i)t}\int_0^t e^{(-2i)t}\int_0^t e^{(1+i)t}(.)\ dtdt. \tag{4.2}$$

Using (2.7), the solution u is given by

$$u=\frac{x}{2i}e^{-(1+i)t}-\frac{1-i}{2i}xe^{-(1+i)t}-\frac{x}{2i}e^{-(1-i)t}+\frac{1+i}{2i}xe^{-(1-i)t}+L_t^{-1}(u_{xx})+L_t^{-1}(xe^{-t})$$

$$u=\frac{x}{2}e^{-(1+i)t}+\frac{x}{2}e^{-(1-i)t}+L_t^{-1}(u_{xx})+L_t^{-1}(xe^{-t}).$$

So we have

$$\sum_{k=0}^{\infty}u_k=\frac{x}{2}e^{-(1+i)t}+\frac{x}{2}e^{-(1-i)t}+L_t^{-1}(xe^{-t})+L_t^{-1}(\sum_{k=0}^{\infty}(u_k)_{xx}).$$

By the decomposition method we get the following recurrence relations

$$\begin{aligned}u_0&=\frac{x}{2}e^{-(1+i)t}+\frac{x}{2}e^{-(1-i)t}+L_t^{-1}(xe^{-t}),\\ u_{k+1}&=L_t^{-1}(\sum_{k=0}^{\infty}(u_k)_{xx})\quad \forall k\geq 0.\end{aligned}$$

$$\begin{aligned}L_t^{-1}(xe^{-t})&=e^{-(1-i)t}\int_0^t e^{(-2i)t}\int_0^t e^{(1+i)t}(xe^{-t})dtdt\\ &=xe^{-(1-i)t}\int_0^t e^{(-2i)t}\int_0^t e^{it}dtdt\\ &=xe^{-(1-i)t}\int_0^t e^{(-2i)t}[\frac{1}{i}e^{it}-\frac{1}{i}]dt\\ &=xe^{-(1-i)t}\int_0^t[\frac{1}{i}e^{-it}-\frac{1}{i}e^{(-2i)t}]dt\\ &=xe^{-(1-i)t}[e^{-it}-\frac{e^{(-2i)t}}{2}-\frac{1}{2}]\\ &=xe^{-t}-\frac{x}{2}e^{-(1+i)t}-\frac{x}{2}e^{-(1-i)t}.\end{aligned}$$

The components of $u(x,t)$ is given by

$$\begin{aligned}u_0&=\frac{x}{2}e^{-(1+i)t}+\frac{x}{2}e^{-(1-i)t}+xe^{-t}-\frac{x}{2}e^{-(1+i)t}-\frac{x}{2}e^{-(1-i)t}\\ &=xe^{-t},\\ u_1&=L_t^{-1}(u_0)_{xx}=0,\\ &\vdots\\ u_k&=0\quad \forall k=1,2,3,....\end{aligned}$$

Therefore the exact solution is

$$u=u_0=xe^{-t}.$$

4.2 Example 2

Consider the linear telegraph equation [11]

$$\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} + 4u = \frac{\partial^2 u}{\partial x^2}, \tag{4.3}$$

with initial conditions

$$u(x,0) = 1 + e^{2x} \qquad , \qquad u_t(x,0) = -2.$$

Here we have $\alpha = \beta$.
Substituting $\alpha = 2$ in Eq.(2.11) we get

$$\Rightarrow L_t(.) = e^{-2t}\frac{\partial^2}{\partial t^2}e^{2t}(.). \tag{4.4}$$

The inverse operator L_t^{-1} is

$$L_t^{-1}(.) = e^{-2t}\int_0^t\int_0^t e^{2t}(.)dtdt. \tag{4.5}$$

Then by Eq.(2.16) we get

$$u = e^{-2t} + e^{2x-2t}(1+2t) + L_t^{-1}\frac{\partial^2 u}{\partial x^2}$$

By the decomposition method we get the following recurrence relations

$$u_0 = e^{-2t} + e^{2x-2t}(1+2t),$$
$$u_{k+1} = L_t^{-1}\frac{\partial^2 u_k}{\partial x^2}, \quad \forall k = 0,1,2,3,...$$

The components of the solution $u(x,t)$ is given by

$$\begin{aligned}
u_0(x,t) =& e^{-2t} + e^{2x-2t}(1+2t),\\
u_1(x,t) =& e^{-2t}\int_0^t\int_0^t e^{2t}(4e^{2x-2t}(1+2t))\ dtdt = e^{2x-2t}\left(\frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right),\\
u_2(x,t) =& e^{-2t}\int_0^t\int_0^t e^{2t}(4e^{2x-2t}\left(\frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right))\ dtdt = e^{2x-2t}\left(\frac{(2t)^4}{4!} + \frac{(2t)^5}{5!}\right),\\
&\vdots\\
u_k(x,t) =& e^{2x-2t}\left(\frac{(2t)^{2k}}{(2k)!} + \frac{(2t)^{2k+1}}{(2k+1)!}\right).
\end{aligned}$$

The solution $u(x,t)$ in a series form is given by

$$\begin{aligned}
u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t)....\\
=& e^{-2t} + e^{2x-2t}\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + ...\right).
\end{aligned}$$

Which gives the exact solution

$$u(x,t) = e^{-2t} + e^{2x-2t}e^{2t} = e^{-2t} + e^{2x}.$$

4.3 Example 3

Consider the telegraph equation [4]

$$u_{tt} + u_t + u = u_{xx} + x^2 + t - 1, \tag{4.6}$$

with initial conditions

$$u(x,0) = x^2 \quad , \quad u_t(x,0) = 1.$$

Here we have $\alpha + \beta = 1$ and $\alpha\beta = 1$,

$$\Rightarrow \quad \alpha = \frac{1}{2} - \frac{i\sqrt{3}}{2} \quad , \quad \beta = \frac{1}{2} + \frac{i\sqrt{3}}{2}.$$

So, by Eq.(2.6) the inverse operator is given by

$$L_t^{-1}(.) = e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \int_0^t e^{(i\sqrt{3})t} \int_0^t e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} (.)dtdt.$$

Substituting α, β and the initial conditions in (2.7) we get

$$u = e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} \left(\frac{1}{i\sqrt{3}} + (\frac{1}{2} - \frac{i\sqrt{3}}{6})x^2 \right) + e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \left(\frac{-1}{i\sqrt{3}} + (\frac{1}{2} + \frac{i\sqrt{3}}{6})x^2 \right) + L_t^{-1}(x^2 + t - 1) + L_t^{-1} u_{xx}$$

$$\begin{aligned} L_t^{-1}(x^2 + t - 1) =& e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \int_0^t e^{(i\sqrt{3})t} \int_0^t e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} \left(x^2 + t - 1\right) dtdt \\ =& x^2 + t - 2 + e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} \left((\frac{-1}{2} + \frac{i\sqrt{3}}{6})x^2 + 1 \right) + e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \left((\frac{-1}{2} - \frac{i\sqrt{3}}{6})x^2 + 1 \right) \end{aligned}$$

$$u = x^2 + t - 2 + e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} \left(\frac{1}{i\sqrt{3}} + 1 \right) + e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \left(\frac{-1}{i\sqrt{3}} + 1 \right) + L_t^{-1} u_{xx}.$$

So by ADM, we have the following recurrence relations

$$\begin{aligned} u_0 &= x^2 + t - 2 + e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} \left(\frac{1}{i\sqrt{3}} + 1 \right) + e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \left(\frac{-1}{i\sqrt{3}} + 1 \right), \\ u_{k+1} &= L_t^{-1} \left(u_k\right)_{xx}, \quad k = 0, 1, 2, 3, ... \end{aligned}$$

Therefore

$$\begin{aligned} u_1 =& L_t^{-1}(u_0)_{xx} = L_t^{-1}(2) = e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t} \left(\frac{-1}{i\sqrt{3}} - 1 \right) + e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t} \left(\frac{1}{i\sqrt{3}} - 1 \right) + 2, \\ u_2 =& L_t^{-1}(u_1)_{xx} = L_t^{-1}(0) = 0, \\ &\vdots \\ u_k =& 0 \quad , k = 2, 3, 4, ... \end{aligned}$$

$$\Rightarrow u = u_0 + u_1 + u_2 + ... = x^2 + t.$$

Which is the exact solution.

4.4 Example 4

We will solve this example first by applying the general case, and then we will solve it by using the suggested transformation technique to see how effective this suggested technique.

Consider the following tow-dimensional telegraph equation [6]

$$u_{tt} + 3u_t + 2u = u_{xx} + u_{yy}, \tag{4.7}$$

with initial conditions

$$u(x, y, 0) = e^{x+y} \quad , \quad u_t(x, y, 0) = -3e^{x+y}.$$

Here we have $\alpha + \beta = 3$, and $\alpha\beta = 2 \quad \Rightarrow \alpha = 1, \quad \beta = 2.$

Substituting α, β and the given initial conditions in (2.7) we get

$$u = e^{x+y}(2e^{-2t} - e^{-t}) + L_t^{-t}[u_{xx} + u_{yy}], \tag{4.8}$$

where

$$L_t^{-t}(.) = e^{-2t}\int_0^t e^t \int_0^t e^t(.)dtdt.$$

So

$$u_0 = e^{x+y}(2e^{-2t} - e^{-t}),$$

and

$$u_{k+1} = L_t^{-t}[(u_k)_{xx} + (u_k)_{yy}].$$

$$\begin{aligned}
u_1 =& e^{x+y-t}(6 - 2t) + e^{x+y-2t}(-6 - 4t),\\
u_2 =& e^{x+y-t}(-2t^2 + 16t - 36) + e^{x+y-2t}(4t^2 + 20t + 36),\\
u_3 =& e^{x+y-t}(-\frac{4}{3}t^3 + 20t^2 - 112t + 240) + e^{x+y-2t}(-\frac{8}{3}t^3 - 28t^2 - 128t - 240),\\
u_4 =& e^{x+y-t}(-\frac{2}{3}t^4 + 16t^3 - 160t^2 + 800t - 1680) + e^{x+y-2t}(\frac{4}{3}t^4 + 24t^3 + 200t^2 + 880t + 1680),\\
u_5 =& e^{x+y-t}(-\frac{4}{15}t^5 + \frac{28}{3}t^4 - 144t^3 + 1232t^2 - 5824t + 12096) + e^{x+y-2t}(-\frac{8}{15}t^5 - \frac{44}{3}t^4 - 192t^3\\
&- 1456t^2 - 6272t - 12096),\\
&\vdots
\end{aligned}$$

Therefor the solution in a series form is
$u = u_1 + u_2 + u_3 + u_4 + ...$

$$\begin{aligned}
u =& -\frac{4}{15}e^{x+y-t}(-\frac{159375}{4} + \frac{38415}{2}t - \frac{8175}{2}t^2 + 485t^3 - \frac{65}{2}t^4 + t^5 + ...)\\
&+ \frac{1}{15}e^{x+y-2t}(-159360 - 82560t - 19200t^2 - 2560t^3 - 200t^4 - 8t^5 + ...).
\end{aligned}$$

Now we will solve this problem by the special case transformation technique .
First we rewrite the equation as

$$\frac{\partial^2 u}{\partial t^2}+2(\frac{3}{2})\frac{\partial u}{\partial t}+(\frac{3}{2})^2 u=\frac{1}{4}u+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

Substituting $\alpha=\frac{3}{2}$ and using the given initial conditions in Eq.(2.11) we get

$$\Rightarrow L_t(.)=e^{-\frac{3}{2}t}\ \frac{\partial^2}{\partial t^2}\ e^{\frac{3}{2}t}(.). \tag{4.9}$$

The inverse operator L_t^{-1} is

$$L_t^{-1}(.)=e^{-\frac{3}{2}t}\int_0^t\int_0^t e^{\frac{3}{2}t}(.)\ dtdt. \tag{4.10}$$

Substituting α,β and the initial conditions in Eq.(2.16) we get

$$u=-3te^{x+y-(\frac{3}{2})t}+e^{x+y-(\frac{3}{2})t}+\frac{3}{2}te^{x+y-(\frac{3}{2})t}+L_t^{-1}\frac{1}{4}u+L_t^{-1}\frac{\partial^2 u}{\partial x^2}+L_t^{-1}\frac{\partial^2 u}{\partial y^2}$$

$u=e^{x+y-(\frac{3}{2})t}[1-\frac{3}{2}t]+L_t^{-1}\frac{1}{4}u+L_t^{-1}\frac{\partial^2 u}{\partial x^2}+L_t^{-1}\frac{\partial^2 u}{\partial y^2}.$

By the decomposition method we have the following recurrence relations

$$\begin{aligned}
u_0&=e^{x+y-(\frac{3}{2})t}[1-\frac{3}{2}t],\\
u_k&=L_t^{-1}\frac{1}{4}(u_{k-1})+L_t^{-1}(\frac{\partial^2 u_{k-1}}{\partial x^2})\ +L_t^{-1}(\frac{\partial^2 u_{k-1}}{\partial y^2})\\
&=\frac{9}{4}L_t^{-1}(u_{k-1}),\forall\ k=0,1,2,3,\dots
\end{aligned}$$

Therefore the components of the solution $u(x,y,t)$ is

$$\begin{aligned}
u_0&=e^{x+y-(\frac{3}{2})t}(1-\frac{3}{2}t),\\
u_1&=\frac{9}{4}L_t^{-1}u_0=\frac{9}{4}L_t^{-1}(e^{x+y-\frac{3}{2}t}(1-\frac{3}{2}t))=\frac{9}{4}e^{-\frac{3}{2}t}\int_0^t\int_0^t e^{\frac{3}{2}t}e^{x+y-\frac{3}{2}t}(1-\frac{3}{2}t)dtdt\\
&=\frac{4}{9}e^{x+y-\frac{3}{2}t}\left(\frac{t^2}{2!}-\frac{3t^3}{2*3!}\right)=e^{x+y-\frac{3}{2}t}\left(\frac{(-\frac{3t}{2})^2}{2!}-\frac{(-\frac{3t}{2})^3}{3!}\right),\\
u_2&=e^{x+y-\frac{3}{2}t}\left(\frac{(-\frac{3t}{2})^4}{4!}-\frac{(-\frac{3t}{2})^5}{5!}\right),\\
&\vdots\\
u_k&=e^{x+y-\frac{3}{2}t}\left(\frac{(-\frac{3t}{2})^{2k}}{(2k)!}-\frac{(-\frac{3t}{2})^{(2k+1)}}{(2k+1)!}\right).
\end{aligned}$$

The solution in a series form is given by

$$u(x,y,t)=e^{x+y-\frac{3}{2}t}[1-(-\frac{3t}{2})+\frac{(-\frac{3t}{2})^2}{2!}-\frac{(-\frac{3t}{2})^3}{3!}+\frac{(-\frac{3t}{2})^4}{4!}-\frac{(-\frac{3t}{2})^5}{5!}+...].$$

Which gives the exact solution

$$u(x,y,t) = e^{x+y-\frac{3}{2}t}e^{-\frac{3}{2}t} = e^{x+y-3t}.$$

As illustrated by this example, when the equation is converted to the special case form, the MADM iterations become easier to compute and more effectively converge to the exact solution, in comparison to the general case.

We computed the absolute error of each series for this example at various points to compare their convergence rates. The results are presented in Table 1, where the series are truncated at six terms (k=0 to k=5).

4.5 Example 5

Consider the telegraph equation [3]

$$u_{tt} + u_t - u = u_{xx}, \quad x \in [0,1], \tag{4.11}$$

with initial conditions

$$u(x,0) = \sin x \quad , \quad u_t(x,0) = -\sin x, \quad t \geq 0.$$

First we rewrite Eq.(4.11) as follows

$$u_{tt} + 2(\frac{1}{2})u_t + \frac{1}{4}u = \frac{5}{4}u + u_{xx}. \tag{4.12}$$

Substituting $\alpha = \frac{1}{2}$ in Eq.(3.3) then

$$L_t(.) = e^{-\frac{t}{2}}\frac{\partial^2}{\partial t^2}e^{\frac{t}{2}}(.),$$

and the inverse operator becomes

$$L_t^{-1}(.) = e^{-\frac{t}{2}}\int_0^t\int_0^t e^{\frac{t}{2}}(.)\ dtdt.$$

Then by Eq.(3.5) we get

$$\begin{aligned} u(x,t) &= -\sin x\ te^{-\frac{t}{2}} + \sin x\ e^{-\frac{t}{2}} + \frac{1}{2}\sin x\ te^{-\frac{t}{2}} + L_t^{-1}(\frac{5}{4}u + u_{xx}) \\ &= \sin x\ e^{-\frac{t}{2}}[1 - \frac{t}{2}] + L_t^{-1}(\frac{5}{4}u + u_{xx}). \end{aligned}$$

$$\sum_{k=0}^{\infty} u_k(x,t) = \sin x\ e^{-\frac{t}{2}}[1 - \frac{t}{2}] + L_t^{-1}(\frac{5}{4}\sum_{k=0}^{\infty} u_k + \sum_{k=0}^{\infty}(u_k)_{xx})$$

By the decomposition method

$$u_0(x,t) = \sin x\ e^{-\frac{t}{2}}(1 - \frac{t}{2}),$$

$$u_k(x,t) = L_t^{-1}(\frac{5}{4}u_{k-1} + (u_{k-1})_{xx}).$$

So the components of the solution u(x,t) is given by

$$\begin{aligned}
u_0(x,t) &= \sin x\ e^{-\frac{t}{2}}(1-\frac{t}{2}),\\
u_1(x,t) &= L_t^{-1}(\frac{5}{4}u_0+(u_0)_{xx}) = \frac{1}{4}L_t^{-1}(\sin x\ e^{-\frac{t}{2}}[1-\frac{t}{2}])\\
&= \frac{1}{4}\sin x\ e^{-\frac{t}{2}}\left(\frac{t^2}{2!}-\frac{t^3}{2*3!}\right) = \sin x\ e^{-\frac{t}{2}}\left(\frac{(\frac{t}{2})^2}{2!}-\frac{(\frac{t}{2})^3}{3!}\right),\\
u_2(x,t) &= \frac{1}{4}\frac{1}{4}\sin x\ e^{-\frac{t}{2}}\left(\frac{t^4}{4!}-\frac{t^5}{2*5!}\right) = \sin x\ e^{-\frac{t}{2}}\left(\frac{(\frac{t}{2})^4}{4!}-\frac{(\frac{t}{2})^5}{5!}\right),\\
&\vdots\\
u_k(x,t) &= \sin x\ e^{-\frac{t}{2}}\left(\frac{(\frac{t}{2})^{(2k)}}{(2k)!}-\frac{(\frac{t}{2})^{(2k+1)}}{(2k+1)!}\right).
\end{aligned}$$

The solution in a series form is given by

$$\begin{aligned}
u(x,t) &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\dots\\
&= \sin x\ e^{-\frac{t}{2}}\left(1-\frac{t}{2}+\frac{(\frac{t}{2})^2}{2!}-\frac{(\frac{t}{2})^3}{3!}+\frac{(\frac{t}{2})^4}{4!}-\frac{(\frac{t}{2})^5}{5!}+\dots\right).
\end{aligned}$$

Which gives the exact solution

$$\begin{aligned}
u(x,t) &= \sin x\ e^{-\frac{t}{2}}e^{-\frac{t}{2}}\\
&= \sin x\ e^{-t}.
\end{aligned}$$

4.6 Example 6

Consider the nonlinear telegraph equation [16]

$$u_{tt}+2u_t = u_{xx}-u^2+e^{2x-4t}-e^{x-2t}, \tag{4.13}$$

with initial conditions

$$u(x,0)=e^x, \quad u_t(x,0)=-2e^x.$$

We have

$$\alpha=2, \quad \beta=0.$$

Substituting α,β in 2.4 results in

$$L_t(.)=e^{-2t}\frac{\partial}{\partial t}e^{2t}\frac{\partial}{\partial t}(.).$$

By 2.6 the inverse integral operator is

$$L_t^{-1}=\int_0^t e^{-2t}\int_0^t e^{2t}(.)dtdt.$$

Therefore by substituting the given initial conditions in 2.7, the solution is given by

$$u(x,t)=e^{x-2t}+L_t^{-1}(e^{2x-4t}-e^{x-2t})+L_t^{-1}u_{xx}-L_t^{-1}u^2.$$

By the decomposition method the recursive relations are

$$\begin{aligned}
u_0 =& e^{x-2t}+L_t^{-1}(e^{2x-4t}-e^{x-2t}),\\
u_{k+1} =& L_t^{-1}(u_k)_{xx}-L_t^{-1}(A_k), \quad k+0,1,2,...
\end{aligned}$$

where A_k are the Adomian polynomials for the nonlinear term u^2, and they can be calculated as follows[1]

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0 u_1, \\ A_2 &= 2u_0 u_2 + u_1^2, \\ &\vdots \end{aligned}$$

The components u_n of the series solution can be determined as follows

$$\begin{aligned} u_0 &= e^{x-2t} + \int_0^t e^{-2t} \int_0^t e^{2t}(e^{2x-4t} - e^{x-2t})\, dtdt \\ &= \frac{t}{2}e^{x-2t} + \frac{5}{4}e^{x-2t} + \frac{1}{8}e^{2x-4t} - \frac{1}{4}e^{2x-2t} + \frac{1}{8}e^{2x} - \frac{1}{4}e^{2x}, \\ u_1 &= L_t^{-1}(u_0)_{xx} - L_t^{-1}(A_0) = \int_0^t e^{-2t} \int_0^t e^{2t}((u_0)_{xx} - u_0^2)\, dtdt, \\ u_2 &= L_t^{-1}(u_1)_{xx} - L_t^{-1}(A_1) = \int_0^t e^{-2t} \int_0^t e^{2t}((u_1)_{xx} - 2u_0 u_1)\, dtdt, \\ &\vdots \end{aligned}$$

The solution in series form is

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k = u_0 + u_1 + u_2 + ... \\ &= \frac{t}{2}e^{x-2t} + \frac{5}{4}e^{x-2t} + \frac{1}{8}e^{2x-4t} - \frac{1}{4}e^{2x-2t} + \frac{1}{8}e^{2x} - \frac{1}{4}e^{2x} + ..., \end{aligned}$$

where the exact solution is

$$u(x,t) = e^{x-2t}.$$

The absolute error of this Example 6 is displayed in Table **??**. It was computed with three terms of the solution i.e $u_0 + u_1 + u_2$, at $x = 0.01$ for different values of t.

Table 1. Comparison between exact solution $u = e^{x+y-3t}$ of Example 4 and the solution by MADM and SMADM at different values of x, y and t using 6 iterations(i.e k=5)

x,y	t	solution by MADM	solution by SMADM	EXACT SOLUTION	AE by MADM	AE by SMADM
0	0.1	0.7408180000	0.7408182207	0.7408182207	2.207×10^{-7}	0
	0.3	0.4065730000	0.4065696597	0.4065696597	3.3403×10^{-6}	0
	0.5	0.2231300000	0.2231301600	0.2231301601	1.601×10^{-7}	1×10^{-10}
0.5	0.1	2.013750000	2.013752708	2.013752707	2.707×10^{-6}	1×10^{-9}
	0.3	1.105170000	1.105170918	1.105170918	9.18×10^{-7}	0
	0.5	0.606540000	0.6065306596	0.6065306597	9.3403×10^{-6}	1×10^{-10}
1	0.1	5.473970000	5.473947392	5.473947392	2.2608×10^{-5}	0
	0.3	3.004170000	3.004166024	3.004166024	3.976×10^{-6}	0
	0.5	1.648730000	1.648721270	1.648721271	8.729×10^{-6}	1×10^{-9}

Table 2. Comparison between the exact solution $u(x,t) = e^{x-2t}$ of Example 6 and the solution obtained by our MADM with three terms of solution $u(x,t) = u_0 + u_1 + u_2$

t	Exact solution at x=0.01	MADM solution at x=0.01	AE
0.1	0.8269591339	0.8269591350	1.1×10^{-9}
0.2	0.6770568745	0.6770570197	1.452×10^{-7}
0.3	0.5543272847	0.5543296764	2.3917×10^{-6}
0.4	0.4538447953	0.4538606685	15.87×10^{-6}
0.5	0.3715766910	0.3716419421	65.25×10^{-6}
0.6	0.3042212641	0.3044206706	199.41×10^{-6}
0.7	0.2490753046	0.2495734460	498.14×10^{-6}
0.8	0.2039256117	0.2050013396	1.08×10^{-3}
0.9	0.1669601697	0.1690413190	2.08×10^{-3}
1	0.1366954254	0.1403909208	3.70×10^{-3}

5 Conclusions

In conclusion, the Modified Adomian Decomposition Method (MADM) with inverse differential operator has been demonstrated to effectively solve both linear and nonlinear telegraph equations. In some examples, exact solutions were derived, while in others, very good approximations were obtained. The suggested transformation technique proved effective, with series derived using this technique converging faster to the exact solution. Overall, the MADM is a promising method for solving the telegraph equation due to its ease of implementation and efficiency in finding both approximate and analytical solutions.

Competing Interests

Authors have declared that no competing interests exist.

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