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*Research Article*

# Gevrey Asymptotics for Logarithmic-Type Solutions to Singularly Perturbed Problems with Nonlocal Nonlinearities

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We investigate a family of nonlinear partial differential equations which are singularly perturbed in a complex parameter $\epsilon$ and singular in a complex time variable $t$ at the origin. These equations combine differential operators of Fuchsian type in time $t$ and space derivatives on horizontal strips in the complex plane with a nonlocal operator acting on the parameter $\epsilon$ known as the formal monodromy around 0. Their coefficients and forcing terms comprise polynomial and logarithmic-type functions in time and are bounded holomorphic in space. A set of logarithmic-type solutions are shaped by means of Laplace transforms relatively to $t$ and $\epsilon$ and Fourier integrals in space. Furthermore, a formal logarithmic-type solution is modeled which represents the common asymptotic expansion of the Gevrey type of the genuine solutions with respect to $\epsilon$ on bounded sectors at the origin.

## 1. Introduction

In this paper, we examine a family of singularly perturbed nonlinear partial differential equations modeled as

$$\begin{aligned} Q(\partial_z)u(t,z,\epsilon) &= (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon) \\ &\quad + P(t,z,\epsilon,t\partial_t,\partial_z)u(t,z,\epsilon) + f(t,z,\epsilon) \\ &\quad + H\Big(\log(\epsilon t), z, \epsilon, \{P_j(\partial_z)u(t,z,\epsilon)\}_{j\in J_1}, \{Q_j(\partial_z)\gamma_\epsilon^* u(t,z,\epsilon)\}_{j\in J_2}\Big), \end{aligned} \tag{1}$$

for vanishing initial data $u(0,z,\epsilon)\equiv 0$. The constants $d_D$, $\delta_D \geq 1$ are natural numbers and $Q(X)$, $R_D(X)$, and $P_j(X)$ for $j\in J_1$ and $Q_j(X)$ for $j\in J_2$, where $J_1$ and $J_2$ are two finite subsets of the positive integers $\mathbb{N}^*$, which stand for polynomials with complex coefficients. The linear differential operator $P(t,z,\epsilon,t\partial_t,\partial_z)$ depends analytically in a perturbation parameter $\epsilon$ on a disc $D_{\epsilon_0}$ with radius $\epsilon_0>0$ centered at 0 and relies polynomially in the complex time $t$ and holomorphically with respect to the space variable $z$ on a horizontal strip framed as $H_\beta=\{z\in\mathbb{C}/|\text{Im}(z)|<\beta\}$ in $\mathbb{C}$, for some given width $2\beta>0$. The forcing term $f(t,z,\epsilon)$ is a map of the logarithmic type represented as a sum

$$f(t,z,\epsilon)=f_1(t,z,\epsilon)+f_2(t,z,\epsilon)\log(\epsilon t), \tag{2}$$

where $f_j(t,z,\epsilon)$ and $j=1,2$ are polynomials in $t$, with holomorphic coefficients in $z$ on $H_\beta$ and in $\epsilon$ on $D_{\epsilon_0}$. The map $H(v_0,z,\epsilon,\{v_j\}_{j\in J_1},\{w_j\}_{j\in J_2})$ is a specific polynomial of degree at most 2 in its arguments $v_0$, $\{v_j\}_{j\in J_1}$, and $\{w_j\}_{j\in J_2}$, which relies holomorphically in $z$ on $H_\beta$ and in $\epsilon$ on $D_{\epsilon_0}$. The precise shape of $H$ is framed in (36).

The main objective of the work (depicted in Theorem 24 of Subsection 8.2) is the construction of a set of logarithmic-type holomorphic solutions to the nonlinear Equation (1), and the analysis of their asymptotic power series expansions in the small parameter $\epsilon$ on sectors in $\mathbb{C}^*$ centered at 0.

The nonlinear term $H$ of (1) involves not only powers of $P_j(\partial_z)u(t,z,\epsilon)$ and $j\in J_1$, but also powers of derivatives of $\gamma_\epsilon^* u(t,z,\epsilon)$ where $\gamma_\epsilon^*$ is a nonlocal operator acting on $u(t,z,\epsilon)$ which represents the so-called monodromy operator around 0 relatively to $\varepsilon$. In the literature, the concept of formal monodromy around a point $a$ in $\mathbb{C}$ appears in the construction of formal fundamental solutions to linear systems of differential equations with a so-called irregular singularity at the given point $a$, known as the Levelt-Turrittin theorem, see [1]. It asserts that a differential system of the form

$$x^r Y'(x) = A(x)Y(x), \tag{3}$$

for analytic coefficient matrix $A(x) \in M_n(\mathbb{C})\{x\}$ near 0 with $n \geq 1$, for an integer $r \geq 2$, with an irregular singularity at 0, possesses a formal fundamental solution with the shape

$$\widehat{Y}(x) = \widehat{P}\left(x^{1/e}\right)x^C \exp\left(\varphi\left(x^{1/e}\right)\right), \tag{4}$$

for some well-chosen integer $e \geq 1$, where $\widehat{P}(y) \in \mathrm{GL}_n(\mathbb{C}[[y]][1/y])$ is a formal meromorphic invertible matrix, $\varphi(x^{1/e})$ is a diagonal matrix whose coefficients are polynomials in $x^{-1/e}$ with complex coefficients and $C \in M_n(\mathbb{C})$ is related to the so-called formal monodromy matrix $M \in \mathrm{GL}_n(\mathbb{C})$ by the formula $M = \exp(2\pi i C)$. It is worth remarking that this formal monodromy matrix extends in the formal settings the so-called monodromy matrix that appears in the representation of fundamental matrix solutions to systems (3) with a regular singularity of the form

$$Y(x) = H(x)x^E, \tag{5}$$

where $H$ is an invertible matrix with meromorphic coefficients near 0, for a matrix $E$ giving rise to the monodromy matrix $N \in \mathrm{GL}_n(\mathbb{C})$ by means of $N = \exp(2\pi i E)$. The matrix $N$ is obtained as an analytic continuation of the fundamental matrix solution $Y(x)$ along a simple loop $\gamma$ going counterclockwise around the origin 0 with base point $x$ by means of the identity

$$\gamma^* Y(x) = Y(x)N, \tag{6}$$

where $\gamma^* Y$ denotes the analytic continuation along $\gamma$, see [2]. In the same manner as the analytic continuation operator $\gamma^*$ acting on analytic functions, a formal monodromy operator $\gamma^*$ acting on various spaces and rings (such as the so-called Picard-Vessiot rings) through the formulas $\gamma^*(z^\lambda) = e^{2i\pi\lambda}z^\lambda$ for complex numbers $\lambda \in \mathbb{C}$ and $\gamma^*(l) = l + 2i\pi$ where $l$ is the symbol for the Log function has been introduced and studied from an abstract and algebraic point of view in the textbook [1].

In our context, the action of the formal monodromy $\gamma_\epsilon^*$ on $u(t,z,\epsilon)$ can be reformulated as a shift mapping on angles $\theta \mapsto \theta + 2\pi$ in polar coordinates by means of the change of functions

$$u(t,z,\epsilon) = v(t,z,r,\theta), \tag{7}$$

for $\epsilon = re^{\sqrt{-1}\theta}$, with radius $r > 0$ and angle $\theta \in \mathbb{R}$, through the formula

$$\gamma_\epsilon^* u(t,z,\epsilon) = v(t,z,r,\theta + 2\pi). \tag{8}$$

In this way, the main Equation (1) can be recast as some nonlinear mixed type partial difference-differential equation for the map $v(t,z,r,\theta)$. This class of equations has become the object of many investigations these last years and possesses numerous applications to engineering problems and biology, see for instance the introductive book [3].

In the context of singularly perturbed differential equations, most of the papers in the literature are devoted to advanced or delayed equations of the form

$$\epsilon\partial_t x(t,\epsilon) = f(t,\epsilon,x(t,\epsilon),x(t \pm \delta,\epsilon)), \tag{9}$$

for some vector-valued function $f$, where $\varepsilon$ stands for a small positive parameter and where $\delta > 0$ is some fixed constant. Some abstract convergence results and historical background can be found in [4]. The construction of solutions $x(t,\epsilon)$ having asymptotic expansions of the form

$$x(t,\epsilon) = \sum_{l=0}^{n-1} x_l(t)\epsilon^l + R_n(t,\epsilon), \tag{10}$$

with error-bound estimates for the remainder $R_n$, $n \geq 1$, is extensively discussed for instance in the recent textbook [5] for the linear setting and in the paper [6] for some nonlinear cases.

In the particular situation of nonlinear difference equations in the complex domain with the shape

$$y(z+1) = F(z,y(z)), \tag{11}$$

for $\mathbb{C}^n$ – valued analytic maps $F$ in a neighborhood of $(\infty, y_0)$ for some $y_0 \in \mathbb{C}^n$, we notice that important results concerning asymptotic features of their solutions as $z$ tends to infinity have been obtained by several authors, see [7–9].

In the framework of singularly perturbed partial differential equations, we refer to some interesting works in the case involving delay operators such as [10–13] or entailing nonlinearities with nonlocal operators of integral type related to the famous Schrödinger equation in physics such as [14–17].

We highlight our premise that the main Equation (1) counts in powers of the basic differential operator $t\partial_t$ which is labelled of Fuchsian type. We refer to [18] for many sharp results about Fuchsian ordinary and partial differential equations. However, under the sufficient conditions required on (1) listed in Subsection 2.2, it pans out that (1) will be reduced throughout the work to a coupling of two partial differential equations, stated in (81) and (82), that comprise only powers of the basic differential operator $u_1^{k_1+1}\partial_{u_1}$, for a well-chosen integer $k_1 \geq 1$, of irregular type in a complex variable $u_1$. The definition of irregular-type differential operators is given in the classical textbook [19] in the ordinary differential equation settings displayed in (3) and in the work [20] in the framework of partial differential equations.

In the present contribution, we cook up a set of holomorphic solutions to (1) shaped as logarithmic-type maps that involve Fourier/Laplace transforms. Namely, under the list of requirements which mould (1) and detailed in Subsection 2.3, one can outline

(i) A set of properly selected bounded open sectors $\{\epsilon_p\}_{p\in I_1}$ for some finite set $I_1 \subset \mathbb{N}$ and $\mathcal{T}$ centered at 0

(ii) A family of holomorphic functions $u_p(t,z,\epsilon)$, $p \in I_1$, which conforms to solutions of (1) on the domain $\mathcal{T} \times H_\beta \times \epsilon_p$. Each solution $u_p$, $p \in I_1$, is expressed as a sum

$$u_p(t,z,\epsilon) = u_{1,p}(t,z,\epsilon) + u_{2,p}(t,z,\epsilon)\log(\epsilon t), \tag{12}$$

where each component $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, is represented as a Fourier/Laplace transform

$$u_{j,p}(t,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_p}} \int_{-\infty}^{+\infty} \omega_{j,d_p}(\tau,m,\epsilon) \cdot \exp\left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm, \tag{13}$$

where the commonly named Borel/Fourier map $\omega_{j,d_p}(\tau, m, \epsilon)$ stands for a function

(i) Which is analytic near $\tau = 0$

(ii) Being (at most) of exponential growth of some order $k_1 \geq 1$ on an infinite sector containing the half line $L_{d_p} = [0,+\infty)e^{\sqrt{-1}d_p}$ with respect to $\tau$ for suitable direction $d_p \in \mathbb{R}$

(iii) Continuous and subjected to an exponential decay with respect to $m$ on $\mathbb{R}$

with analytic dependence in $\varepsilon$ on the punctured disc $D_{\epsilon_0} \setminus \{0\}$.

Furthermore, owing to their Laplace integral structure, the components $\{u_{j,p}\}_{p\in I_1}$ own asymptotic expansions of Gevrey type in the parameter $\epsilon$. Indeed, for given $j = 1,2$, all the partial functions $\epsilon \mapsto u_{j,p}(t,z,\epsilon)$, $p \in I_1$, share a common asymptotic formal power series expansion

$$\widehat{\mathbb{G}}_j(\epsilon) = \sum_{n\geq 0} \mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}, \tag{14}$$

on $\epsilon_p$, with bounded holomorphic coefficients $\mathbb{G}_{n,j}$ on $\mathcal{T} \times H_\beta$. These asymptotic expansions turn out to be of Gevrey order $1/k_1$ on every sector $\epsilon_p$, meaning that constants $K_{p,j}$, $M_{p,j} > 0$ can be singled out for which the error bounds

$$\left| u_{j,p}(t,z,\epsilon) - \sum_{n=0}^{N} \mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!} \right| \leq K_{p,j}(M_{p,j})^{N+1}\Gamma\left(1 + \frac{N+1}{k_1}\right)|\epsilon|^{N+1}, \tag{15}$$

hold for all integers $N \geq 0$, all $\epsilon \in \epsilon_p$, uniformly in $t \in \mathcal{T}$ and $z \in H_\beta$. At last, we verify that the formal logarithmic-type expression

$$\widehat{\mathbb{G}}(\epsilon) = \widehat{\mathbb{G}}_1(\epsilon) + \widehat{\mathbb{G}}_2(\epsilon)\log(\epsilon t), \tag{16}$$

itself obeys the main Equation (1).

Throughout the proof of our main result, we show that the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$ of the built-up solutions $u_p$, $p \in I_1$, to (1) turn out to be embedded in a larger family of maps $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, for all integers $0 \leq p \leq \varsigma - 1$ for some integer $\varsigma \geq 2$. These maps are bounded holomorphic on products $\mathcal{T} \times H_\beta \times \epsilon_p$ where $\underline{\epsilon} = \{\epsilon_p\}_{0\leq p\leq\varsigma-1}$ stands for a set of bounded sectors, entailing $\epsilon_p$ for $p \in I_1$, which represents a good covering in $\mathbb{C}^*$ (see Definition 18). Each map $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, is modeled as a rescaled version of a bounded holomorphic map $(u_1,z) \mapsto U_{j,d_p}(u_1,z,\epsilon)$ through

$$u_{j,p}(t,z,\epsilon) = U_{j,d_p}(\epsilon t,z,\epsilon), \tag{17}$$

on domains $U_{1,d_p} \times H_\beta$ for any fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, where $U_{1,d_p}$ are bounded sectors bisected by the direction $d_p$, depicted in Definition 19 of the work. The set of maps $\{U_{2,d_p}\}_{0\leq p\leq\varsigma-1}$ is shown to solve a specific nonlinear partial differential equation with coefficients that are polynomial in $u_1$, holomorphic with respect to $\epsilon$ on $D_{\epsilon_0}$ and relatively to $z$ on $H_\beta$ displayed in (66). The set of maps $\{U_{1,d_p}\}_{0\leq p\leq\varsigma-1}$ conforms a particular nonlinear partial differential equation stated in (67) whose coefficients and forcing term bring in not only polynomials in $u_1$ and holomorphic dependence relatively to $\epsilon$ on $D_{\epsilon_0}$ and to $z$ on $H_\beta$ but also polynomial reliance on the maps $\{U_{2,d_p}\}_{0\leq p\leq\varsigma-1}$ and their derivatives with respect to $u_1$ and $z$. In this sense, the maps $\{U_{j,d_p}\}_{0\leq p\leq\varsigma-1}$, $j = 1,2$, solve a coupling of nonlinear partial differential equations. The asymptotic property for the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$, of $u_p(t,z,\epsilon)$ stems from sharp exponential bound estimates for the differences of neighboring maps $u_{j,p+1} - u_{j,p}$ reached in Proposition 21, for which a classical statement for the existence of asymptotic expansions of the Gevrey type can be applied, see Subsection 8.1.

In this work, as mentioned above, we restrict ourselves to quadratic nonlinearities. Besides, they are chosen in a way to respect the natural triangular structure of the systems of partial differential equations satisfied by the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$ stated in (256) and (257), which stems from the linear part of (1). It means that its resolution is reduced to the study of a coupling of two equations which comprise one single equation satisfied by $u_{2,p}(t,z,\epsilon)$ and a second equation for $u_{1,p}(t,z,\epsilon)$ with coefficients and forcing term that involve $u_{2,p}(t,z,\epsilon)$.

From a computational or numerical viewpoint, two cogent merits of our approach consist in the facts that

(i) The coefficients $\mathbb{G}_{n,j}(t,z)$ and $n \geq 0$ of the formal expansions $\widehat{\mathbb{G}}_j(\epsilon)$, $j=1,2$ satisfy simple and explicit recursion relations (stated in (262) and (270))

(ii) The well-known least-term truncation method applies in our context for the analytic components $u_{j,p}$ of our solutions $u_p$ to (1) since the expansions involved (15) are of Gevrey type, leading to error bounds with exponential accuracy. Namely, one can find constants $L_j, C_j, M_j > 0$, $j=1,2$, such that the piecewise holomorphic map

$$\widehat{\mathbb{G}}_{j,L_j}(\epsilon) := \sum_{n<L_j/|\epsilon|^{k_1}} \mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}, \tag{18}$$

satisfies

$$\left|u_{j,p}(t,z,\epsilon) - \widehat{\mathbb{G}}_{j,L_j}(\epsilon)\right| \leq C_j \exp\left(-\frac{M_j}{|\epsilon|^{k_1}}\right), \tag{19}$$

for all $\epsilon \in \epsilon_p$, uniformly in $t \in \mathcal{T}$ and $z \in H_\beta$. For a reference about this truncation method, we quote the textbook [21], Subsection 4.5.

The major drawback of our proposal is that strong regularity (namely holomorphy) is assumed on the solutions $u_p$ we build up and on the coefficients and forcing term of our main Equation (1). Besides, we need to work with a complex parameter $\epsilon \in \mathbb{C}^*$. Most of the recent studies in the domain of singularly perturbed advanced-delay differential or partial differential equations are obtained under weaker requirements and involve a positive real parameter $\epsilon > 0$. For an overview on recent numerical approaches developed for problems both singularly perturbed and advanced-delay and for statements on their error bounds, we mention [22, 23].

The approach developed in this work can be extended to the construction of both formal and genuine holomorphic solutions to comparable problems as (1) with higher-order logarithmic terms

$$u(t,z,\epsilon) = \sum_{j=0}^{n} u_j(t,z,\epsilon)(\log(\epsilon t))^j, \tag{20}$$

for $n \geq 2$, for suitable nonlinear terms and forcing terms chosen properly in a similar way as the ones in the present work. We focus on the complete description for the case $n=1$ for the sake of simplicity in order to give the readers a clear idea of the main purpose of the study and avoiding cumbersome notations and computations.

Logarithmic-type solutions have been extensively studied in the framework of nonlinear partial differential equations with so-called Fuchsian type and described in the Chapter 8 of the textbook by Gérard and Tahara [18]. Namely, these authors consider nonlinear partial differential equations with the shape

$$(t\partial_t)^m u(t,x) = F\left(t,x,\left\{(t\partial_t)^j \partial_x^\alpha u(t,x)\right\}_{(j,\alpha)\in I_m}\right), \tag{21}$$

where $I_m = \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n / j + |\alpha| \leq m, j < m\}$ for some integers $m, n \geq 1$, for analytic maps $F(t,x,Z)$ near the origin in $\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^{\mathrm{card}\,(I_m)}$. Under conditions of nonresonance of the characteristic exponents at $x=0$ combined with some Poincaré condition on the characteristic polynomial associated to (21), they have described the holomorphic solutions to (21) with at most polynomial growth in $t$ on bounded sectors centered at 0, for $x$ near the origin in $\mathbb{C}^n$ as the maps written in the form of a convergent logarithmic type expression

$$u(t,x) = u_0(t,x) + \sum_{(i,j,k)\in J_m} \varphi_{i,j,k}(x) t^{i+\sum_{l=1}^{\mu} j_l \rho_l(x)} (\log(t))^k, \tag{22}$$

for $J_m = \{(i,j,k) \in \mathbb{N}\times\mathbb{N}^\mu\times\mathbb{N} / i + 2m|j| \geq k + 2m, |j| \geq 1\}$ where

(i) $u_0$ stands for convergent power series near the origin

(ii) $\rho_l(x)$ and $1 \leq l \leq \mu$ are the characteristic exponents with positive real parts at $x=0$

(iii) $\varphi_{i,j,k}(x)$ are holomorphic coefficients near $x=0$

In the case of so-called equations of irregular type or non-Fuchsian type, in which our present work falls, fewer results are known and represent a favourable breeding ground for upcoming research. Nonetheless, in that trend, we mention the remarkable recent general result [24] obtained by Tahara. This work extends a paper by Yamazawa which treats linear partial differential equations, see [25]. Therein, the author examines nonlinear partial differential equations

$$F\left(t,x,\left\{(t\partial_t)^j \partial_x^\alpha u(t,x)\right\}_{(j,\alpha)\in L_m}\right) = 0, \tag{23}$$

with $L_m = \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^K / j + |\alpha| \leq m\}$, for some integers $m, K \geq 1$, which possess a formal series (which is divergent in the generic situation)

$$\hat{u}(t,x) = \sum_{n\geq 1} u_n(t,x), \tag{24}$$

solution where each term $u_n$, $n \geq 1$, is analytic with respect to $t$ on some appropriate bounded sector $S$ centered at 0 in $\mathbb{C}$ and holomorphic near 0 relatively to $x$ on some disc $D_R$ in $\mathbb{C}^K$. In general, these expressions $u_n$ might involve combinations of functions of the form $t^{\lambda(x)}$ for holomorphic maps $\lambda$, powers of $t$ and $\log(t)$, and analytic functions with respect to $x$ on $D_R$. The author introduced a so-called Newton polygon associated to Equation (23) along the formal solution $\hat{u}(t,x)$. In the case this Newton polygon possesses $p \geq 1$ slopes and under some additional technical requirements, the author builds up a new formal solution

$$\hat{w}(t,x) = \sum_{n\geq 1} w_n(t,x), \quad (25)$$

to (23) which is subjected to the next two features

(i) The formal series $\hat{u}$ and $\hat{w}$ are asymptotically equivalent in the sense that for any $A > 0$, there exists $N_0 \geq 1$, such that

$$\sup_{x\in D_R} \left| (t\partial_t)^j \partial_x^\alpha (\hat{u}_N - \hat{w}_N) \right| \leq C|t|^A, \quad (26)$$

for all $t \in S$, $j + |\alpha| \leq m$, some constant $C > 0$, any $N \geq N_0$, where $\hat{u}_N$ and $\hat{w}_N$ denote the partial sums of the $N$ first terms of $\hat{u}$ and $\hat{w}$.

The formal series $\hat{w}$ is multisummable on $S$ with respect to $t$, uniformly in $x$ on $D_R$, in a sense that enhances the classical multisummability process described in [19] and gives rise to a genuine holomorphic solution $w(t,x)$ of (23) on $S \times D_R$ crafted as iterated analytic acceleration operators and Laplace integral of some Borel transform of $\hat{w}$.

Thereupon, it turns out that $w(t,x)$ admits $\hat{u}(t,x)$ as an asymptotic expansion as $t$ tends to 0 on $S$ in the sense that for any $A > 0$, there exists $N_0 \geq 1$ such that

$$\sup_{x\in D_R} |w(t,x) - \hat{u}_N(t,x)| \leq C|t|^A, \quad (27)$$

for all $t \in S$, some constant $C > 0$, any $N \geq N_0$.

At last, in the linear setting, some general results reaching beyond the structure of logarithmic-type solutions have been achieved. Namely, for Cauchy problems

$$a(x,D)u = v, D_{x_0}^h u_{|x_0=0} = 0, \quad 0 \leq h < m, \quad (28)$$

involving linear differential operators $a(x,D)$ of order $m \geq 1$ with holomorphic coefficients in $x = (x_j)_{0\leq j\leq n}$ in $\mathbb{C}^{n+1}$, existence and uniqueness results for so-called ramified solutions around certain characteristic hypersurfaces $K$ in $\mathbb{C}^{n+1}$, provided that $v$ is ramified around $K$, have been obtained by several authors, see [26–28].

## 2. Layout of the Main Equation

*2.1. Formal Monodromy around the Origin.* In this subsection, we define the notion of a formal monodromy operator around the origin acting on different classes of objects. Following the description of abstract formal monodromy operator as stated in Subsection 3.2 of [1], we first provide a definition of formal monodromy acting on logarithmic type expressions involving formal power series with coefficients in Banach spaces.

*Definition 1.* Let $\mathcal{T}$ be a bounded open sector centered at 0 in $\mathbb{C}^*$ and let

$$H_{\beta'} = \left\{ \frac{z \in \mathbb{C}}{|\text{Im}(z)|} < \beta' \right\}, \quad (29)$$

be a strip with width $2\beta' > 0$. We denote $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})$ the Banach space of bounded holomorphic functions on $\mathcal{T} \times H_{\beta'}$ equipped with the sup norm and we set $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})[[\epsilon]]$ as the vector space of all formal series

$$\hat{a}(t,z,\epsilon) = \sum_{n\geq 0} a_n(t,z)\epsilon^n, \quad (30)$$

with coefficients belonging to $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})$. Let $\hat{u}_1(t,z,\epsilon)$ and $\hat{u}_2(t,z,\epsilon)$ be two elements of $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})[[\epsilon]]$; we set the formal logarithmic type expression

$$\hat{u}(t,z,\epsilon) = \hat{u}_1(t,z,\epsilon) + \hat{u}_2(t,z,\epsilon)\log(\epsilon t), \quad (31)$$

where $\log(x)$ stands for the principal value of the logarithm of a complex number $x \in \mathbb{C}^*$.

We define the formal monodromy operator around 0 relatively to $\epsilon$, denoted $\gamma_\epsilon^*$ as acting on $\hat{u}$ by means of

$$\gamma_\epsilon^* \hat{u}(t,z,\epsilon) = \hat{u}_1(t,z,\epsilon) + 2\pi\sqrt{-1}\hat{u}_2(t,z,\epsilon) + \hat{u}_2(t,z,\epsilon)\log(\epsilon t). \quad (32)$$

The next definition of formal monodromy extends the concept of a monodromy operator around 0 acting on analytic functions on a punctured neighborhood of 0 as analytic continuation along a simple loop around the origin as described in [2], Section 16.

*Definition 2.* Let $\mathcal{T}, \epsilon$ be bounded open sectors centered at 0 in $\mathbb{C}$ and $H_{\beta'}$ be a strip defined by (29). We set $\mathcal{O}_b(\mathcal{T} \times H_{\beta'} \times \epsilon)$ as the Banach space of bounded holomorphic functions on $\mathcal{T} \times H_{\beta'} \times \epsilon$ endowed with the sup norm. Let $u_1(t,z,\epsilon), u_2(t,z,\epsilon)$ be two elements of $\mathcal{O}_b(\mathcal{T} \times H_{\beta'} \times \epsilon)$. We set

$$u(t,z,\epsilon) = u_1(t,z,\epsilon) + u_2(t,z,\epsilon)\log(\epsilon t), \quad (33)$$

that represents a holomorphic function for all $(t,z,\epsilon) \in \mathcal{T} \times H_{\beta'} \times \epsilon$ with $\epsilon t \notin (-\infty, 0]$. The formal monodromy operator around 0 relatively to $\epsilon$ denoted $\gamma_\epsilon^*$ acts on $u$ through the formula

$$\gamma_\epsilon^* u(t,z,\epsilon) = u_1(t,z,\epsilon) + 2\pi\sqrt{-1}u_2(t,z,\epsilon) + u_2(t,z,\epsilon)\log(\epsilon t), \quad (34)$$

Notice that if $u_1$ and $u_2$ are holomorphic on a full-punctured disc centered at 0 relatively to $\epsilon$, the formal monodromy $\gamma_\epsilon^*$ given above coincides with the analytic continuation along a simple loop skirting counterclockwise the origin 0 with base point $\epsilon$.

We observe that each components $\widehat{u}_1$,$\widehat{u}_2$ of (31) (resp. $u_1$, $u_2$ of (33)) can be expressed by means of $\widehat{u}$ and $\gamma_\epsilon^*\widehat{u}$ (resp. $u$ and $\gamma_\epsilon^* u$) through the formulas

$$\widehat{u}_2(t,z,\epsilon) = \frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\widehat{u}(t,z,\epsilon),$$
$$\widehat{u}_1(t,z,\epsilon) = \widehat{u}(t,z,\epsilon) - \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})\widehat{u}(t,z,\epsilon)\right]\log(\epsilon t), \tag{35}$$

$$u_2(t,z,\epsilon) = \frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon),$$
$$u_1(t,z,\epsilon) = u(t,z,\epsilon) - \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\log(\epsilon t), \tag{36}$$

where id represents the identity operator acting on $\mathcal{O}_b(\mathcal{T} \times H_{\beta'})[[\epsilon]]$ in (35) and on $\mathcal{O}_b(\mathcal{T} \times H_{\beta'} \times \epsilon)$ in (36).

*2.2. Outline of the Main Problem.* The principal problem under study in this work is shaped as follows:

$$\begin{aligned}
&Q(\partial_z)u(t,z,\epsilon)\\
&\quad = (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon)\\
&\qquad + \sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)u(t,z,\epsilon)\\
&\qquad + f(t,z,\epsilon) + c_1(z,\epsilon)\\
&\qquad \cdot \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\log(\epsilon t) + b_1(z,\epsilon)\\
&\qquad \cdot \left[u(t,z,\epsilon) - \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\log(\epsilon t)\right]\\
&\qquad + b_2(z,\epsilon)\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\\
&\qquad + c_{Q_1Q_2}Q_1(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\\
&\qquad \times Q_2(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\\
&\qquad \times \log(\epsilon t) + c_{P_1P_2}P_1(\partial_z)\\
&\qquad \cdot \left[u(t,z,\epsilon) - \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\log(\epsilon t)\right]\\
&\qquad \times P_2(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\\
&\qquad + c_{P_3P_4}P_3(\partial_z)\left[u(t,z,\epsilon) - \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\right.\\
&\qquad \left.\cdot \log(\epsilon t)\right] \times P_4(\partial_z)\left[u(t,z,\epsilon)\right.\\
&\qquad \left. - \left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\log(\epsilon t)\right]\\
&\qquad + c_{P_5P_6}P_5(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right]\\
&\qquad \times P_6(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^* - \mathrm{id})u(t,z,\epsilon)\right],
\end{aligned} \tag{37}$$

for vanishing initial data $u(0,z,\epsilon) \equiv 0$. On the way in reaching our main result Theorem 24, we need to impose a list of constraints on the building blocks of (37). Namely,

(i) The numbers $D \geq 2$, $d_D, \delta_D \geq 1$, and $\Delta_l, d_l, \delta_l \geq 1$, $1 \leq l \leq D-1$ are integers that are subjected to the next restrictions

(1) We assume the existence of an integer $k_1 \geq 1$ with

$$d_D = \delta_D k_1. \tag{38}$$

(2) The inequalities

$$d_l > \delta_l k_1, \tag{39}$$

hold for all $1 \leq l \leq D-1$.

(3) The bounds

$$k_1\delta_D - 1 \geq k_1\delta_l, \tag{40}$$

are asked for all $1 \leq l \leq D-1$.

(4) The lower estimates

$$\Delta_l \geq 1 + \delta_l k_1, \tag{41}$$

are mandatory for all $1 \leq l \leq D-1$.

(ii) The constants $c_{Q_1Q_2}$, $c_{P_jP_{j+1}}$, and $j = 1,3,5$ are non-vanishing complex numbers that are chosen close enough to 0 (the precise constraints that these numbers are asked to obey are stated later on in the work, see Sections 5 and 6.

(iii) The maps $Q(X), R_l(X)$, $l = 1,\cdots,D$, and $Q_1(X)$, $Q_2(X)$ along with $P_j(X)$, $1 \leq j \leq 6$ are polynomials with complex coefficients. We require that

$$\deg(R_l) \leq \deg(R_D), \tag{42}$$

for $1 \leq l \leq D-1$ and

$$\deg(R_D) \geq \deg(Q_1), \quad \deg(R_D) \geq \deg(Q_2), \quad \deg(R_D) \geq \deg(P_j), \tag{43}$$

for $1 \leq j \leq 6$. Furthermore, we require the existence of an unbounded sectorial annulus

$$S_{Q,R_D} = \left\{ z \in \mathbb{C}^* / r_{Q,R_D} < |z|, \left|\arg(z) - d_{Q,R_D}\right| \leq \eta_{Q,R_D} \right\}, \tag{44}$$

with bisecting direction $d_{Q,R_D} \in \mathbb{R}$, aperture $\eta_{Q,R_D} > 0$, and inner radius $r_{Q,R_D} > 0$ (prescribed later in the work), for which the next inclusion

$$\left\{ \frac{Q\left(\sqrt{-1}m\right)/R_D\left(\sqrt{-1}m\right)}{m \in \mathbb{R}} \right\} \subset S_{Q,R_D}, \tag{45}$$

occurs.

The forcing term $f(t,z,\epsilon)$ is built up in the next manner. In its construction, we make use of Banach spaces that have been introduced in [29] and brought into play in several works by the author.

*Definition 3.* Let $\beta, \mu \in \mathbb{R}$. We set $E_{(\beta,\mu)}$ as the vector space of continuous functions $h : \mathbb{R} \longrightarrow \mathbb{C}$ such that

$$\|h(m)\|_{(\beta,\mu)} = \sup_{m \in \mathbb{R}} (1+|m|)^{\mu} \exp(\beta|m|)|h(m)|, \tag{46}$$

is finite. The space $E_{(\beta,\mu)}$ endowed with the norm $\|.\|_{(\beta,\mu)}$ becomes a Banach space.

The forcing term is written as a sum

$$f(t,z,\epsilon) = f_1(t,z,\epsilon) + f_2(t,z,\epsilon)\log(\epsilon t), \tag{47}$$

where the components $f_1, f_2$ are set up as follows. Let $J_1$, $J_2 \subset \mathbb{N}^*$ be finite subsets of the positive integers. For $l=1$, 2 and $j_l \in J_l$, we denote $m \mapsto \mathscr{F}_{l,j_l}(m,\epsilon)$ maps that

(i) Appertain to the Banach space $E_{(\beta,\mu)}$ for some $\beta > 0$ and

$$\mu > \deg(R_l) + 1, \mu > \max(\deg(Q_1)+1, \deg(Q_2)+1), \quad \mu > \max_{1 \leq j \leq 6}\left\{\deg\left(P_j\right)+1\right\}, \tag{48}$$

for all $1 \leq l \leq D-1$.

(ii) Rely analytically on $\epsilon$ on some disc $D_{\epsilon_0}$ with radius $\epsilon_0 > 0$ for which constants $\mathrm{F}_{l,j_l,\epsilon_0} > 0$ exist such that

$$\sup_{\epsilon \in D_{\epsilon_0}} \left\| \mathscr{F}_{l,j_l}(m,\epsilon) \right\|_{(\beta,\mu)} \leq \mathrm{F}_{l,j_l,\epsilon_0}. \tag{49}$$

For $l=1,2$, let us introduce the polynomials in the variable $\tau$ with coefficients in $E_{(\beta,\mu)}$,

$$\mathscr{F}_l(\tau,m,\epsilon) = \sum_{j_l \in J_l} \mathscr{F}_{l,j_l}(m,\epsilon)\tau^{j_l}, \tag{50}$$

and set the integral representations

$$F_l(T,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \mathscr{F}_l(\tau,m,\epsilon) \cdot \exp\left(-\left(\frac{\tau}{T}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm, \tag{51}$$

where $L_{d_1} = [0,+\infty)e^{\sqrt{-1}d_1}$ is a half line in direction $d_1 \in \mathbb{R}$ that relies on $T$ under the constraint $\cos(k_1(d_1 - \arg(T))) > 0$. We observe that $F_1$ and $F_2$ are polynomials in $T$ and can be expanded in the form

$$F_l(T,z,\epsilon) = \sum_{j_l \in J_l} F_{l,j_l}(z,\epsilon)\Gamma\left(\frac{j_l}{k_1}\right)T^{j_l}, \tag{52}$$

where $\Gamma(x)$ stands for the Gamma function, for coefficients given by the inverse Fourier integral expressions

$$F_{l,j_l}(z,\epsilon) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \mathscr{F}_{l,j_l}(m,\epsilon) e^{\sqrt{-1}zm} dm, \tag{53}$$

that are bounded holomorphic on the product $H_{\beta'} \times D_{\epsilon_0}$, for any given $0 < \beta' < \beta$, where $H_{\beta'}$ is the horizontal strip given by (29), for $l=1,2$. Eventually, we set the components

$$f_l(t,z,\epsilon) = F_l(\epsilon t, z, \epsilon), \tag{54}$$

of (47) as a time-rescaled version of $F_l$, for $l=1,2$, that represent bounded holomorphic functions on $\mathbb{C} \times H_{\beta'} \times D_{\epsilon_0}$.

The coefficients $a_l(z,\epsilon)$, $1 \leq l \leq D-1$, $c_1(z,\epsilon)$, and $b_j(z,\epsilon)$, $j=1,2$ are manufactured as follows. Let $m \mapsto A_l(m,\epsilon)$, $1 \leq l \leq D-1$, $m \mapsto C_1(m,\epsilon)$, and $m \mapsto B_j(m,\epsilon)$, $j=1,2$, be maps that

(i) Belong to the Banach space $E_{(\beta,\mu)}$, for the real numbers $\beta > 0$ and $\mu > 1$ given above

(ii) That depend analytically in $\epsilon$ on $D_{\epsilon_0}$ and for which positive constants $\mathrm{A}_{l,\epsilon_0}$, $1 \le l \le D-1$, $\mathrm{C}_{1,\epsilon_0}$, and $\mathrm{B}_{j,\epsilon_0}$, $j = 1, 2$ can be singled out with

$$\sup_{\epsilon \in D_{\epsilon_0}} \|A_l(m,\epsilon)\|_{(\beta,\mu)} \le \mathrm{A}_{l,\epsilon_0},$$
$$\sup_{\epsilon \in D_{\epsilon_0}} \|C_1(m,\epsilon)\|_{(\beta,\mu)} \le \mathrm{C}_{1,\epsilon_0},$$
$$\sup_{\epsilon \in D_{\epsilon_0}} \left\|B_j(m,\epsilon)\right\|_{(\beta,\mu)} \le \mathrm{B}_{j,\epsilon_0}. \tag{55}$$

We set

$$\begin{aligned} a_l(z,\epsilon) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m,\epsilon) e^{\sqrt{-1}zm} dm, \\ c_1(z,\epsilon) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} C_1(m,\epsilon) e^{\sqrt{-1}zm} dm, \\ b_j(z,\epsilon) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} B_j(m,\epsilon) e^{\sqrt{-1}zm} dm, \end{aligned} \tag{56}$$

for $1 \le l \le D-1$, $j = 1, 2$. The maps $a_l$, $1 \le l \le D-1$, $c_1$, and $b_j$, $j = 1, 2$ represent bounded holomorphic maps on the product $H_{\beta'} \times D_{\epsilon_0}$, for any prescribed $0 < \beta' < \beta$.

## 3. Couplings of Related Initial Value Problems

*3.1. A Coupling of Associated Partial Differential Equations.* We seek for solutions $u(t,z,\epsilon)$ to our main Equation (37) in the form

$$u(t,z,\epsilon) = U(\epsilon t, \log(\epsilon t), z, \epsilon), \tag{57}$$

for some expression $U(u_1,u_2,z,\epsilon)$ in the four independent variables $u_1, u_2, z, \epsilon$. We furthermore assume that $U$ is an affine map relatively to $u_2$ meaning that $U$ is the polynomial of degree at most one in $u_2$.

We first disclose an equation fulfilled by $U(u_1,u_2,z,\epsilon)$ provided that $u(t,z,\epsilon)$ solves (37) given by (62). According to the usual chain rule applied at a formal level at this stage of the work, we first observe that

$$\begin{aligned} t\partial_t u(t,z,\epsilon) &= t[\partial_t(\epsilon t)]\left(\partial_{u_1} U\right)(\epsilon t, \log(\epsilon t), z, \epsilon) \\ &\quad + [t\partial_t(\log(\epsilon t))]\left(\partial_{u_2} U\right)(\epsilon t, \log(\epsilon t), z, \epsilon) \\ &= \left[\left(u_1 \partial_{u_1} + \partial_{u_2}\right) U\right](\epsilon t, \log(\epsilon t), z, \epsilon). \end{aligned} \tag{58}$$

Besides, owing to the assumption that $U$ is affine in $u_2$, we can decompose $U$ in the form

$$U(u_1,u_2,z,\epsilon) = U_1(u_1,z,\epsilon) + U_2(u_1,z,\epsilon)u_2, \tag{59}$$

for some expressions $U_j(u_1,z,\epsilon)$, $j = 1, 2$. If one sets

$$u_j(t,z,\epsilon) = U_j(\epsilon t, z, \epsilon), \tag{60}$$

for $j = 1, 2$, through (57), one arrives at the next expansion of $u$,

$$u(t,z,\epsilon) = u_1(t,z,\epsilon) + u_2(t,z,\epsilon) \log(\epsilon t). \tag{61}$$

As a result, in view of formulas (35) and (36) together with identity (58) and definitions (54) and (60), we check that $u(t,z,\epsilon)$ formally solves Equation (37) if the expression $U(u_1,u_2,z,\epsilon)$ is subjected to the next equation

$$\begin{aligned} &Q(\partial_z)U(u_1,u_2,z,\epsilon) \\ &\quad = u_1^{d_D}\left(u_1\partial_{u_1} + \partial_{u_2}\right)^{\delta_D} R_D(\partial_z) U(u_1,u_2,z,\epsilon) \\ &\qquad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon)\left(u_1\partial_{u_1} + \partial_{u_2}\right)^{\delta_l} R_l(\partial_z) U(u_1,u_2,z,\epsilon) \\ &\qquad + F_1(u_1,z,\epsilon) + F_2(u_1,z,\epsilon)u_2 + c_1(z,\epsilon)U_2(u_1,z,\epsilon)u_2 \\ &\qquad + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon) \\ &\qquad + c_{Q_1Q_2}[Q_1(\partial_z)U_2(u_1,z,\epsilon)] \times [Q_2(\partial_z)U_2(u_1,z,\epsilon)]u_2 \\ &\qquad + c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)] \times [P_2(\partial_z)U_2(u_1,z,\epsilon)] \\ &\qquad + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)] \times [P_4(\partial_z)U_1(u_1,z,\epsilon)] \\ &\qquad + c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)] \times [P_6(\partial_z)U_2(u_1,z,\epsilon)]. \end{aligned} \tag{62}$$

In the next step, we derive some coupling of partial differential equations that the components $U_1$ and $U_2$ are asked to fulfill and displayed in (66) and (67).

Owing to the fact that the operators $u_1\partial_{u_1}$ and $\partial_{u_2}$ commute to each other, the binomial formula helps us to rewrite (62) in the form

$$\begin{aligned} &Q(\partial_z)U(u_1,u_2,z,\epsilon) \\ &\quad = u_1^{d_D}\left[\sum_{p_1+p_2=\delta_D} \frac{\delta_D!}{p_1!p_2!}\left(u_1\partial_{u_1}\right)^{p_1} \partial_{u_2}^{p_2} R_D(\partial_z) U(u_1,u_2,z,\epsilon)\right] \\ &\qquad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon) \\ &\qquad \times \left[\sum_{p_1+p_2=\delta_l} \frac{\delta_l!}{p_1!p_2!}\left(u_1\partial_{u_1}\right)^{p_1} \partial_{u_2}^{p_2} R_l(\partial_z) U(u_1,u_2,z,\epsilon)\right] \\ &\qquad + F_1(u_1,z,\epsilon) + F_2(u_1,z,\epsilon)u_2 + c_1(z,\epsilon)U_2(u_1,z,\epsilon)u_2 \\ &\qquad + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon) \\ &\qquad + c_{Q_1Q_2}[Q_1(\partial_z)U_2(u_1,z,\epsilon)] \times [Q_2(\partial_z)U_2(u_1,z,\epsilon)]u_2 \\ &\qquad + c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)] \times [P_2(\partial_z)U_2(u_1,z,\epsilon)] \\ &\qquad + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)] \times [P_4(\partial_z)U_1(u_1,z,\epsilon)] \\ &\qquad + c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)] \times [P_6(\partial_z)U_2(u_1,z,\epsilon)]. \end{aligned} \tag{63}$$

Besides, from decomposition (59), we observe that

$$\partial_{u_2} U(u_1, u_2, z, \epsilon) = U_2(u_1, z, \epsilon), \quad \partial_{u_2}^{p_2} U(u_1, u_2, z, \epsilon) \equiv 0, \tag{64}$$

whenever $p_2 \geq 2$. We reach the next equation

$$\begin{aligned} &Q(\partial_z)[U_1(u_1,z,\epsilon) + U_2(u_1,z,\epsilon)u_2] \\ &= u_1^{d_D}\left[\left(u_1\partial_{u_1}\right)^{\delta_D} R_D(\partial_z)(U_1(u_1,z,\epsilon) + U_2(u_1,z,\epsilon)u_2)\right. \\ &\quad \left. + \delta_D\left(u_1\partial_{u_1}\right)^{\delta_D-1} R_D(\partial_z)U_2(u_1,z,\epsilon)\right] \\ &\quad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon)\left[\left(u_1\partial_{u_1}\right)^{\delta_l} R_l(\partial_z)(U_1(u_1,z,\epsilon)\right. \\ &\quad \left. + U_2(u_1,z,\epsilon)u_2) + \delta_l\left(u_1\partial_{u_1}\right)^{\delta_l-1} R_l(\partial_z)U_2(u_1,z,\epsilon)\right] \\ &\quad + F_1(u_1,z,\epsilon) + F_2(u_1,z,\epsilon)u_2 + c_1(z,\epsilon)U_2(u_1,z,\epsilon)u_2 \\ &\quad + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon) \\ &\quad + c_{Q_1Q_2}[Q_1(\partial_z)U_2(u_1,z,\varepsilon)] \times [Q_2(\partial_z)U_2(u_1,z,\varepsilon)]u_2 \\ &\quad + c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)] \times [P_2(\partial_z)U_2(u_1,z,\epsilon)] \\ &\quad + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)] \times [P_4(\partial_z)U_1(u_1,z,\epsilon)] \\ &\quad + c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)] \times [P_6(\partial_z)U_2(u_1,z,\epsilon)]. \end{aligned} \tag{65}$$

Finally, by dint of identification of the powers of $u_2$ in the above equality, it turns out that this last Equation (65) holds if the expressions $U_1$ and $U_2$ are asked to satisfy the next coupling of two partial differential equations

$$\begin{aligned} &Q(\partial_z)U_2(u_1,z,\epsilon) \\ &= u_1^{d_D}\left[\left(u_1\partial_{u_1}\right)^{\delta_D} R_D(\partial_z)U_2(u_1,z,\epsilon)\right] \\ &\quad + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon)\left(u_1\partial_{u_1}\right)^{\delta_l} R_l(\partial_z)U_2(u_1,z,\epsilon) \\ &\quad + F_2(u_1,z,\epsilon) + c_1(z,\epsilon)U_2(u_1,z,\epsilon) \\ &\quad + c_{Q_1Q_2}[Q_1(\partial_z)U_2(u_1,z,\epsilon)] \times [Q_2(\partial_z)U_2(u_1,z,\epsilon)], \end{aligned} \tag{66}$$

$$\begin{aligned} &Q(\partial_z)U_1(u_1,z,\epsilon) \\ &= u_1^{d_D}\left[\left(u_1\partial_{u_1}\right)^{\delta_D} R_D(\partial_z)U_1(u_1,z,\epsilon)\right. \\ &\quad \left. + \delta_D\left(u_1\partial_{u_1}\right)^{\delta_D-1} R_D(\partial_z)U_2(u_1,z,\epsilon)\right] \\ &\quad + \sum_{l=1}^{D-1} \varepsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z,\epsilon)\left[\left(u_1\partial_{u_1}\right)^{\delta_l} R_l(\partial_z)U_1(u_1,z,\epsilon)\right. \\ &\quad \left. + \delta_l\left(u_1\partial_{u_1}\right)^{\delta_l-1} R_l(\partial_z)U_2(u_1,z,\epsilon)\right] + F_1(u_1,z,\epsilon) \\ &\quad + b_1(z,\epsilon)U_1(u_1,z,\epsilon) + b_2(z,\epsilon)U_2(u_1,z,\epsilon) \\ &\quad + c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)] \times [P_2(\partial_z)U_2(u_1,z,\epsilon)] \\ &\quad + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)] \times [P_4(\partial_z)U_1(u_1,z,\epsilon)] \\ &\quad + c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)] \times [P_6(\partial_z)U_2(u_1,z,\epsilon)]. \end{aligned} \tag{67}$$

*3.2. A Coupling of Auxiliary Convolution Equations.* We search for solutions to the coupling of partial differential Equations (66) and (67) in the form of a Laplace transform of some order $k_1 \geq 1$ and inverse Fourier integral

$$\begin{aligned} U_{j,d_1}(u_1,z,\epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \omega_{j,d_1}(\tau,m,\epsilon) \\ \cdot \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm, \end{aligned} \tag{68}$$

for $j = 1, 2$, where $L_{d_1} = [0,+\infty)e^{\sqrt{-1}d_1}$ stands for a half line in suitable directions $d_1 \in \mathbb{R}$ which depend on $\tau$ in a way that $\cos(k_1(d_1 - \arg(u_1)))$ remains strictly positive.

Here, we assume that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the so-called Borel-Fourier maps $(\tau, m) \mapsto \omega_{j,d_1}(\tau, m, \epsilon)$, $j = 1, 2$, belong to the Banach space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ for well-chosen constants $\nu, \rho > 0$ and for the prescribed constants $\beta, \mu$ in Subsection 2.2 that is described in the upcoming definition.

*Definition 4.* Let $\epsilon_0, \nu, \beta, \mu, \rho > 0$ be positive real numbers and $k_1 \geq 1$ be an integer. Let $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. We set as $S_{d_1}$ an unbounded sector centered at 0 with bisecting direction $d_1 \in \mathbb{R}$. We denote $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ the vector space of all continuous maps $(\tau, m) \mapsto h(\tau, m)$ on $(S_{d_1} \cup D_\rho) \times \mathbb{R}$, holomorphic w.r.t $\tau$ on $S_{d_1} \cup D_\rho$, such that the norm

$$\begin{aligned} &\|h(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &= \sup_{\tau \in S_{d_1} \cup D_\rho, m \in \mathbb{R}} (1+|m|)^\mu e^{\beta|m|} \frac{|\epsilon|}{|\tau|}\left(1 + \left|\frac{\tau}{\epsilon}\right|^{2k_1}\right) \\ &\quad \cdot \exp\left(-\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right)|h(\tau,m)|, \end{aligned} \tag{69}$$

is finite. The vector space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ equipped with the norm $\|.\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ turns out to be a Banach space.

The main purpose of this subsection is to determine coupling convolution equations for the Borel-Fourier maps $\omega_{j,d_1}$ outlined in (83)–(85). We depart from some features of the Laplace transforms under the action of multiplication by a monomial and differential operators that were already stated and proved in our foregoing work [30], Lemma 6.

**Lemma 5.** *The next identities hold.*

(1) *The action of the differential operator $u_1^{k_1+1}\partial_{u_1}$ on the integral representations $U_{j,d_1}$ is given by*

$$\begin{aligned} &u_1^{k_1+1}\partial_{u_1} U_{j,d_1}(u_1,z,\epsilon) \\ &= \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \left[k_1 \tau^{k_1} \omega_{j,d_1}(\tau,m,\epsilon)\right] \\ &\quad \cdot \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \end{aligned} \tag{70}$$

(2) *Let $m' \geq 1$ be an integer. The multiplication by $u_1^{m'}$ acting on $U_{j,d_1}$ is expressed through*

$$u_1^{m'} U_{j,d_1}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \left[ \frac{\tau^{k_1}}{\Gamma(m'/k_1)} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{(m'/k_1)-1} \cdot \omega_{j,d_1}\left(s^{1/k_1}, m, \epsilon\right) \frac{ds}{s} \right] \times \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \quad (71)$$

(3) *Let $m \mapsto A(m)$ be a map that belongs to $E_{(\beta,\mu)}$. We set*

$$a(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A(m) e^{\sqrt{-1}zm} dm. \quad (72)$$

*The action of multiplication by $a(z)$ on $U_{j,d_1}$ is expressed by means of*

$$a(z) U_{j,d_1}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \left[ \frac{1}{(2\pi)^{1/2}} \cdot \int_{-\infty}^{+\infty} A(m - m_1) \omega_{j,d_1}(\tau, m_1, \epsilon) dm_1 \right] \times \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \quad (73)$$

(4) *Let $H_k(X) \in \mathbb{C}[X]$, $k = 1, 2$, be polynomials. The action of the differential operators $H_k(\partial_z)$ combined with the product of the resulting functions $H_k(\partial_z) U_{j,d_1}$ for $k = 1, 2$, $j = 1, 2$ maps $U_{j,d_1}$ into a Fourier-Laplace transform*

$$\left[H_1(\partial_z) U_{l,d_1}(u_1, z, \epsilon)\right] \times \left[H_2(\partial_z) U_{p,d_1}(u_1, z, \epsilon)\right] = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_1}} \int_{-\infty}^{+\infty} \left[ \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} \cdot H_1\left(\sqrt{-1}(m - m_1)\right) \omega_{l,d_1}\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1, \epsilon\right) \times H_2\left(\sqrt{-1} m_1\right) \omega_{p,d_1}\left(s^{1/k_1}, m_1, \epsilon\right) \frac{1}{(\tau^{k_1} - s) s} ds dm_1 \right] \times \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \quad (74)$$

*for given $1 \leq l, p \leq 2$.*

The next useful lemma already stated in the previous work by Lastra and Malek [31] will show up in the process.

**Lemma 6.** *For all integers $p_1 \geq 1$, positive integers $a_{q,p_1} \geq 1$, for $1 \leq q \leq p_1$ can be singled out such that*

$$\left(u_1 \partial_{u_1}\right)^{p_1} = \sum_{q=1}^{p_1} a_{q,p_1} u_1^q \partial_{u_1}^q, \quad (75)$$

*with $a_{1,p_1} = a_{p_1,p_1} = 1$.*

With the help of this lemma, Equations (66) and (67) can be remodeled in the form

$$\begin{aligned} Q(\partial_z) U_2(u_1, z, \epsilon) &= u_1^{d_D} \left[ \left( \sum_{q=1}^{\delta_D} a_{q,\delta_D} u_1^q \partial_{u_1}^q \right) R_D(\partial_z) U_2(u_1, z, \epsilon) \right] \\ &+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z, \epsilon) \left( \sum_{q=1}^{\delta_l} a_{q,\delta_l} u_1^q \partial_{u_1}^q \right) R_l(\partial_z) U_2(u_1, z, \epsilon) \\ &+ F_2(u_1, z, \epsilon) + c_1(z, \epsilon) U_2(u_1, z, \epsilon) \\ &+ c_{Q_1 Q_2} [Q_1(\partial_z) U_2(u_1, z, \epsilon)] \times [Q_2(\partial_z) U_2(u_1, z, \epsilon)], \end{aligned} \quad (76)$$

$$\begin{aligned} Q(\partial_z) U_1(u_1, z, \epsilon) &= u_1^{d_D} \left[ \left( \sum_{q=1}^{\delta_D} a_{q,\delta_D} u_1^q \partial_{u_1}^q \right) R_D(\partial_z) U_1(u_1, z, \epsilon) \right. \\ &\left. + \delta_D \left( \sum_{q=1}^{\delta_D - 1} a_{q,\delta_D - 1} u_1^q \partial_{u_1}^q \right) R_D(\partial_z) U_2(u_1, z, \epsilon) \right] \\ &+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} u_1^{d_l} a_l(z, \epsilon) \\ &\cdot \left[ \left( \sum_{q=1}^{\delta_l} a_{q,\delta_l} u_1^q \partial_{u_1}^q \right) R_l(\partial_z) U_1(u_1, z, \epsilon) \right. \\ &\left. + \delta_l \left( \sum_{q=1}^{\delta_l - 1} a_{q,\delta_l - 1} u_1^q \partial_{u_1}^q \right) R_l(\partial_z) U_2(u_1, z, \epsilon) \right] \\ &+ F_1(u_1, z, \epsilon) + b_1(z, \epsilon) U_1(u_1, z, \epsilon) \\ &+ b_2(z, \epsilon) U_2(u_1, z, \epsilon) + c_{P_1 P_2} [P_1(\partial_z) U_1(u_1, z, \epsilon)] \\ &\times [P_2(\partial_z) U_2(u_1, z, \epsilon)] + c_{P_3 P_4} [P_3(\partial_z) U_1(u_1, z, \epsilon)] \\ &\times [P_4(\partial_z) U_1(u_1, z, \epsilon)] + c_{P_5 P_6} [P_5(\partial_z) U_2(u_1, z, \epsilon)] \\ &\times [P_6(\partial_z) U_2(u_1, z, \epsilon)]. \end{aligned} \quad (77)$$

The upcoming identity will also be called into play for the derivation of the coupling convolution equations. This technical formula was introduced in the work [32].

**Lemma 7.** *Let $k_1, \delta \geq 1$ be integers. Real numbers $A_{\delta,p}$, for $1 \leq p \leq \delta - 1$ can be found such that*

$$u_1^{\delta(k_1+1)}\partial_{u_1}^{\delta} = \left(u_1^{k_1+1}\partial_{u_1}\right)^{\delta} + \sum_{1\leq p\leq \delta-1} A_{\delta,p} u_1^{k_1(\delta-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}, \tag{78}$$

*holds, where we assume by convention that the sum $\sum_{1\leq p\leq\delta-1}[..]$ vanishes for $\delta = 1$.*

Owing to the assumption (38), the splitting

$$d_D + q = q(k_1+1) + d_{D,q}, \tag{79}$$

holds for suitable integers $d_{D,q} \geq 1$, provided that $1 \leq q \leq \delta_D - 1$. Furthermore, under the constraint (39), the decomposition

$$d_l + q = q(k_1+1) + d_{l,q}, \tag{80}$$

occurs for well-chosen integers $d_{l,q} \geq 1$, as long as $1 \leq l \leq D-1$ and $1 \leq q \leq \delta_l$.

Ultimately, by means of the above two relations (79) and (80), Lemma 7 can be applied in order to rewrite both Equations (76) and (77), only with the help of the basic irregular differential operator $u_1^{k_1+1}\partial_{u_1}$. Namely,

$$\begin{aligned}
&Q(\partial_z)U_2(u_1,z,\epsilon)\\
&= \left(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D} u_1^{d_{D,q}}\left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{q}\right.\right.\\
&\quad \left.+ \sum_{1\leq p\leq q-1} A_{q,p} u_1^{k_1(q-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right] R_D(\partial_z)U_2(u_1,z,\epsilon))\\
&\quad + \left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{\delta_D} + \sum_{1\leq p\leq \delta_D-1} A_{\delta_D,p} u_1^{k_1(\delta_D-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right]\\
&\quad \cdot R_D(\partial_z)U_2(u_1,z,\epsilon) + \left(\sum_{l=1}^{D-1} \epsilon^{\Delta_l-d_l} a_l(z,\epsilon)\right.\\
&\quad \times \left[\sum_{q=1}^{\delta_l} a_{q,\delta_l} u_1^{d_{l,q}}\left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{q} + \sum_{1\leq p\leq q-1} A_{q,p} u_1^{k_1(q-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right]\right.\\
&\quad \left.\left.\cdot R_l(\partial_z)U_2(u_1,z,\epsilon)\right]\right) + F_2(u_1,z,\epsilon) + c_1(z,\epsilon)U_2(u_1,z,\epsilon)\\
&\quad + c_{Q_1Q_2}[Q_1(\partial_z)U_2(u_1,z,\epsilon)] \times [Q_2(\partial_z)U_2(u_1,z,\epsilon)],
\end{aligned} \tag{81}$$

together with

$$\begin{aligned}
&Q(\partial_z)U_1(u_1,z,\epsilon)\\
&= \left(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D} u_1^{d_{D,q}}\left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{q}\right.\right.\\
&\quad \left.\left.+ \sum_{1\leq p\leq q-1} A_{q,p} u_1^{k_1(q-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right] R_D(\partial_z)U_1(u_1,z,\epsilon)\right)\\
&\quad + \left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{\delta_D} + \sum_{1\leq p\leq \delta_D-1} A_{\delta_D,p} u_1^{k_1(\delta_D-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right]\\
&\quad \cdot R_D(\partial_z)U_1(u_1,z,\epsilon) + \delta_D \sum_{q=1}^{\delta_D-1} a_{q,\delta_D-1} u_1^{d_{D,q}}\\
&\quad \cdot \left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{q} + \sum_{1\leq p\leq q-1} A_{q,p} u_1^{k_1(q-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right]\\
&\quad \cdot R_D(\partial_z)U_2(u_1,z,\epsilon) + \left(\sum_{l=1}^{D-1} \epsilon^{\Delta_l-d_l} a_l(z,\epsilon)\right.\\
&\quad \times \left[\sum_{q=1}^{\delta_l} a_{q,\delta_l} u_1^{d_{l,q}}\left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{q}\right.\right.\\
&\quad \left.+ \sum_{1\leq p\leq q-1} A_{q,p} u_1^{k_1(q-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right] R_l(\partial_z)U_1(u_1,z,\epsilon)\\
&\quad + \delta_l \sum_{q=1}^{\delta_l-1} a_{q,\delta_l-1} u_1^{d_{l,q}}\left[\left(u_1^{k_1+1}\partial_{u_1}\right)^{q}\right.\\
&\quad \left.\left.\left.+ \sum_{1\leq p\leq q-1} A_{q,p} u_1^{k_1(q-p)}\left(u_1^{k_1+1}\partial_{u_1}\right)^{p}\right] R_l(\partial_z)U_2(u_1,z,\epsilon)\right]\right)\\
&\quad + F_1(u_1,z,\epsilon) + b_1(z,\epsilon)U_1(u_1,z,\epsilon)\\
&\quad + b_2(z,\epsilon)U_2(u_1,z,\epsilon) + c_{P_1P_2}[P_1(\partial_z)U_1(u_1,z,\epsilon)]\\
&\quad \times [P_2(\partial_z)U_2(u_1,z,\epsilon)] + c_{P_3P_4}[P_3(\partial_z)U_1(u_1,z,\epsilon)]\\
&\quad \times [P_4(\partial_z)U_1(u_1,z,\epsilon)] + c_{P_5P_6}[P_5(\partial_z)U_2(u_1,z,\epsilon)]\\
&\quad \times [P_6(\partial_z)U_2(u_1,z,\epsilon)].
\end{aligned} \tag{82}$$

On the ground of the identities disclosed in Lemma 5, this hindmost coupling of Equations (81) and (82) allows us to reach the next statement.

The maps $U_{j,d_1}(u_1,z,\epsilon)$, $j = 1, 2$, displayed in (68) solve the closing coupling (81) and (82) if the Borel maps $\omega_{j,d_1}(\tau,m,\epsilon)$, $j = 1, 2$, fulfill the next coupling of convolution equations

$$\begin{aligned}
&Q\left(\sqrt{-1}m\right)\omega_{2,d_1}(\tau,m,\epsilon)\\
&= \left(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}\left[\frac{\tau^{k_1}}{\Gamma(d_{D,q}/k_1)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1}\right.\right.\\
&\quad \cdot \left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^{q}\omega_{2,d_1}\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\\
&\quad + \sum_{1\leq p\leq q-1} A_{q,p}\frac{\tau^{k_1}}{\Gamma(d_{D,q}+k_1(q-p)/k_1)}\\
&\quad \cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1}\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^{p}\\
&\quad \left.\left.\cdot \omega_{2,d_1}\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\right] \times R_D\left(\sqrt{-1}m\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\Bigg[\left(k_1\tau^{k_1}\right)^{\delta_D}\omega_{2,d_1}(\tau,m,\epsilon)\\
&+\sum_{1\le p\le\delta_D-1}A_{\delta_D,p}\frac{\tau^{k_1}}{\Gamma(k_1(\delta_D-p)/k_1)}\\
&\cdot\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(k_1(\delta_D-p)/k_1)-1}\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^p\\
&\cdot\omega_{2,d_1}\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\Bigg]\times R_D\left(\sqrt{-1}m\right)\\
&+\sum_{l=1}^{D-1}\varepsilon^{\Delta_l-d_l}\Bigg[\sum_{q=1}^{\delta_l}a_{q,\delta_l}\Bigg[\frac{\tau^{k_1}}{\Gamma\left(d_{l,q}/k_1\right)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{l,q}/k_1\right)-1}\\
&\cdot\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\varepsilon)\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^q\\
&\times R_l\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\\
&+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)}\\
&\cdot\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1}\\
&\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^pR_l\left(\sqrt{-1}m_1\right)\\
&\cdot\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\Bigg]\Bigg]+\mathcal{F}_2(\tau,m,\epsilon)\\
&+\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}C_1(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1\\
&+c_{Q_1Q_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}Q_1\left(\sqrt{-1}(m-m_1)\right)\omega_{2,d_1}\\
&\cdot\left(\left(\tau^{k_1}-s\right)^{1/k_1},m-m_1,\epsilon\right)\times Q_2\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\\
&\cdot\left(s^{1/k_1},m_1,\epsilon\right)\frac{1}{\left(\tau^{k_1}-s\right)s}dsdm_1,
\end{aligned}\tag{83}
$$

along with

$$
\begin{aligned}
&Q\left(\sqrt{-1}m\right)\omega_{1,d_1}(\tau,m,\epsilon)\\
&\quad=\Bigg(\sum_{q=1}^{\delta_D-1}a_{q,\delta_D}\Bigg[\frac{\tau^{k_1}}{\Gamma\left(d_{D,q}/k_1\right)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{D,q}/k_1\right)-1}\\
&\qquad\cdot\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^q\omega_{1,d_1}\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\\
&\qquad+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)}\\
&\qquad\cdot\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{D,q}+k_1(q-p)/k_1\right)-1}\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^p\omega_{1,d_1}\\
&\qquad\cdot\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\Bigg]\times R_D\left(\sqrt{-1}m\right)\Bigg)\\
&\qquad+\Bigg[\left(k_1\tau^{k_1}\right)^{\delta_D}R_D\left(\sqrt{-1}m\right)\omega_{1,d_1}(\tau,m,\epsilon)\\
&\qquad+\sum_{1\le p\le\delta_D-1}A_{\delta_D,p}\frac{\tau^{k_1}}{\Gamma(k_1(\delta_D-p)/k_1)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(k_1(\delta_D-p)/k_1)-1}\\
&\qquad\cdot\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^p\omega_{1,d_1}\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\times R_D\left(\sqrt{-1}m\right)\Bigg]\\
&\qquad+\Bigg(\delta_D\sum_{q=1}^{\delta_D-1}a_{q,\delta_D-1}\Bigg[\frac{\tau^{k_1}}{\Gamma\left(d_{D,q}/k_1\right)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{D,q}/k_1\right)-1}\\
&\qquad\cdot\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^q\omega_{2,d_1}\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\\
&\qquad+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)}\\
&\qquad\cdot\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{D,q}+k_1(q-p)/k_1\right)-1}\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^p\omega_{2,d_1}\\
&\qquad\cdot\left(s^{1/k_1},m,\epsilon\right)\frac{ds}{s}\Bigg]\times R_D\left(\sqrt{-1}m\right)\Bigg)+\sum_{l=1}^{D-1}\varepsilon^{\Delta_l-d_l}\\
&\qquad\cdot\Bigg[\Bigg(\sum_{q=1}^{\delta_l}a_{q,\delta_l}\Bigg[\frac{\tau^{k_1}}{\Gamma\left(d_{l,q}/k_1\right)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{l,q}/k_1\right)-1}\frac{1}{(2\pi)^{1/2}}\\
&\qquad\cdot\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^q\\
&\qquad\times R_l\left(\sqrt{-1}m_1\right)\omega_{1,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\\
&\qquad+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)}\\
&\qquad\cdot\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1}\\
&\qquad\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^p\\
&\qquad\cdot R_l\left(\sqrt{-1}m_1\right)\omega_{1,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\Bigg]\Bigg)\\
&\qquad+\Bigg(\delta_l\sum_{q=1}^{\delta_l-1}a_{q,\delta_l-1}\Bigg[\frac{\tau^{k_1}}{\Gamma\left(d_{l,q}/k_1\right)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{l,q}/k_1\right)-1}\frac{1}{(2\pi)^{1/2}}\\
&\qquad\cdot\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^qR_l\left(\sqrt{-1}m_1\right)\\
&\qquad\times\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\\
&\qquad+\sum_{1\le p\le q-1}A_{q,p}\frac{\tau^{k_1}}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)}\\
&\qquad\cdot\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1}\times\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)
\end{aligned}
$$

$$\cdot\left(k_1\left(s^{1/k_1}\right)^{k_1}\right)^{p} R_l\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\Bigg]\Bigg)\Bigg]$$
$$+\mathscr{A}(\tau,m,\epsilon), \tag{84}$$

where

$$\begin{aligned}\mathscr{A}(\tau,m,\epsilon) &:= \mathscr{F}_1(\tau,m,\epsilon)+\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}B_1(m-m_1,\epsilon)\omega_{1,d_1}\\ &\cdot(\tau,m_1,\epsilon)dm_1+\frac{1}{(2\pi)^{1/2}}\\ &\cdot\int_{-\infty}^{+\infty}B_2(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1\\ &+c_{P_1P_2}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}P_1\left(\sqrt{-1}(m-m_1)\right)\\ &\cdot\omega_{1,d_1}\left(\left(\tau^{k_1}-s\right)^{1/k_1},m-m_1,\epsilon\right)\\ &\times P_2\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\\ &\cdot\frac{1}{\left(\tau^{k_1}-s\right)s}dsdm_1+c_{P_3P_4}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\\ &\cdot\int_0^{\tau^{k_1}}P_3\left(\sqrt{-1}(m-m_1)\right)\omega_{1,d_1}\\ &\cdot\left(\left(\tau^{k_1}-s\right)^{1/k_1},m-m_1,\epsilon\right)\\ &\times P_4\left(\sqrt{-1}m_1\right)\omega_{1,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\\ &\cdot\frac{1}{\left(\tau^{k_1}-s\right)s}dsdm_1+c_{P_5P_6}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\\ &\cdot\int_0^{\tau^{k_1}}P_5\left(\sqrt{-1}(m-m_1)\right)\omega_{2,d_1}\\ &\cdot\left(\left(\tau^{k_1}-s\right)^{1/k_1},m-m_1,\epsilon\right)\\ &\times P_6\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\\ &\cdot\frac{1}{\left(\tau^{k_1}-s\right)s}dsdm_1.\end{aligned} \tag{85}$$

## 4. Linear and Bilinear Convolution Operators Acting on Banach Spaces

In this section, we examine continuity properties of several linear and bilinear convolutions operators that are applied on the Banach spaces given in Definition 4 and that unfold in the above coupled Equations (83)–(85),.

**Proposition 8.** *Let $\gamma_1\geq 0$, $\gamma_3\geq -1$ be integers and set $\gamma_2\in\mathbb{R}$. Let $S_{d_1}$ be an unbounded sector centered at 0 with bisecting direction $d_1\in\mathbb{R}$ and fix $\rho>0$ as some positive real number. Let $a_{\gamma_1}(\tau,m)$ be a continuous map on the closure $(\bar{S}_{d_1}\cup\bar{D}_\rho)\times\mathbb{R}$ subjected to the upper bounds*

$$\left|a_{\gamma_1}(\tau,m)\right|\leq\frac{M_{\gamma_1}}{(1+|\tau|)^{\gamma_1}}, \tag{86}$$

*provided that $\tau\in S_{d_1}\cup D_\rho$, all $m\in\mathbb{R}$, for some constant $M_{\gamma_1}>0$. We take for granted that*

$$\gamma_1\geq k_1(\gamma_3+1),\gamma_2>-1,\gamma_2+\gamma_3+\frac{1}{k_1}+1\geq 0. \tag{87}$$

*Then, we can single out a constant $C_1>0$ (relying on $\gamma_j$, $j=1,2,3$, $k_1$, and $\nu$) for which*

$$\begin{aligned}&\left\|a_{\gamma_1}(\tau,m)\tau^{k_1}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\gamma_2}s^{\gamma_3}f\left(s^{1/k_1},m\right)ds\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\ &\quad\leq C_1M_{\gamma_1}|\epsilon|^{k_1(\gamma_2+1)}\|f(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},\end{aligned} \tag{88}$$

*holds as long as $f$ belongs to the Banach space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*

*Proof.* Let $f\in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. By definition, the bounds

$$\begin{aligned}|f(\tau,m)|&\leq\|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\left|\frac{\tau}{\epsilon}\right|\frac{1}{1+|\tau/\epsilon|^{2k_1}}\\ &\quad\cdot\exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right)(1+|m|)^{-\mu}e^{-\beta|m|},\end{aligned} \tag{89}$$

ensue provided that $\tau\in S_{d_1}\cup D_\rho$ and $m\in\mathbb{R}$. According to the assumption (86), the latter bounds warrant the next estimates

$$\begin{aligned}\mathscr{B}(\tau,m)&:=\left|a_{\gamma_1}(\tau,m)\tau^{k_1}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\gamma_2}s^{\gamma_3}f\left(s^{1/k_1},m\right)ds\right|\\ &\leq\frac{M_{\gamma_1}\|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}}|\tau|^{k_1}\int_0^{|\tau|^{k_1}}\left(|\tau|^{k_1}-h\right)^{\gamma_2}h^{\gamma_3}\frac{h^{1/k_1}}{|\epsilon|}\\ &\quad\cdot\frac{1}{1+\left(h^2/|\epsilon|^{2k_1}\right)}\exp\left(\nu\frac{h}{|\epsilon|^{k_1}}\right)dh\\ &\quad\times(1+|m|)^{-\mu}e^{-\beta|m|}\end{aligned} \tag{90}$$

for all $\tau\in S_{d_1}\cup D_\rho$, all $m\in\mathbb{R}$.

We further perform the change of variable $g = h/|\epsilon|^{k_1}$ in the above integral and get

$$\mathscr{B}(\tau, m) \le \frac{M_{\gamma_1} \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}} |\tau|^{k_1} \int_0^{|\tau|^{k_1}/|\epsilon|^{k_1}} \left( \frac{|\tau|^{k_1}}{|\epsilon|^{k_1}} - g \right)^{\gamma_2} \cdot g^{\gamma_3 + 1/k_1} \frac{1}{1+g^2} e^{\nu g} dg \times |\epsilon|^{k_1(\gamma_2+\gamma_3+1)} (1+|m|)^{-\mu} e^{-\beta|m|}, \tag{91}$$

as long as $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$.

We introduce the function

$$G(x) = \int_0^x (x-g)^{\gamma_2} g^{\gamma_3 + |\tau|^{k_1}/|\epsilon|^{k_1}} \frac{1}{1+g^2} e^{\nu g} dg, \tag{92}$$

for all $x \ge 0$. In the next lemma, we uncover upper bounds for $G$ for large values of $x$. □

**Lemma 4.** *The function $G(x)$ is well defined and continuous for all $x \ge 0$. Furthermore, there exists a constant $K_G > 0$ for which*

$$G(x) \le K_G \frac{x^{\gamma_3 + 1/k_1}}{1+x^2} e^{\nu x}, \tag{93}$$

*for all $x \ge 1$.*

*Proof.* We first explain why $G(x)$ is well defined and continuous for $x \ge 0$. Indeed, by means of the change of variable $g = xg_1$ for $0 \le g_1 \le 1$, we can recast $G(x)$ in the form

$$G(x) = x^{\gamma_2+\gamma_3+(1/k_1)+1} \int_0^1 (1-g_1)^{\gamma_2} g_1^{\gamma_3+(1/k_1)} \frac{1}{1+(xg_1)^2} e^{\nu x g_1} dg_1, \tag{94}$$

which is a finite quantity for all $x \ge 0$ and represents a continuous map w.r.t $x$, according to the last inequality of (87).

In order to reach bounds for large $x \ge 1$, we apply a strategy stemming from Proposition 8 in our joint work [33]. Namely, we split $G(x)$ into two pieces,

$$G(x) = G_1(x) + G_2(x), \tag{95}$$

where

$$\begin{aligned} G_1(x) &= \int_0^{x/2} (x-g_1)^{\gamma_2} g^{\gamma_3+1/k_1} \frac{1}{1+g^2} e^{\nu g} dg, \\ G_2(x) &= \int_{x/2}^{x} (x-g)^{\gamma_2} g^{\gamma_3+1/k_1} \frac{1}{1+g^2} e^{\nu g} dg. \end{aligned} \tag{96}$$

We first focus on bounds for $G_1(x)$. Two cases arise.

(i) Assume that $-1 < \gamma_2 \le 0$. In that situation, we observe that $(x-g)^{\gamma_2} \le (x/2)^{\gamma_2}$ provided that $0 \le g \le x/2$, for $x \ge 0$. Therefore, bearing in mind the constraints (75),

$$\begin{aligned} G_1(x) &\le \left(\frac{x}{2}\right)^{\gamma_2} e^{\nu x/2} \int_0^{x/2} g^{\gamma_3+(1/k_1)} dg \\ &= \frac{1}{\gamma_3 + (1/k_1) + 1} (x/2)^{\gamma_2+\gamma_3+(1/k_1)+1} e^{\nu x/2}, \end{aligned} \tag{97}$$

for all $x \ge 0$.

(ii) Suppose that $\gamma_2 > 0$. We check that $(x-g)^{\gamma_2} \le x^{\gamma_2}$ for any $0 \le g \le x/2$. Hence, paying regard to (75),

$$\begin{aligned} G_1(x) &\le x^{\gamma_2} e^{\nu x/2} \int_0^{x/2} g^{\gamma_3+(1/k_1)} dg \\ &= (1/2)^{\gamma_3+(1/k_1)+1} \frac{1}{\gamma_3+(1/k_1)+1} x^{\gamma_2+\gamma_3+(1/k_1)+1} e^{\nu x/2}, \end{aligned} \tag{98}$$

whenever $x \ge 0$.

In the second step, we provide upper estimates for $G_2(x)$. We notice that $1 + g^2 \ge 1 + (x/2)^2$, for $x/2 \le g \le x$. Hence,

$$G_2(x) \le \frac{1}{1+(x/2)^2} \int_{x/2}^{x} (x-g)^{\gamma_2} g^{\gamma_3+(1/k_1)} e^{\nu g} dg \le \frac{\tilde{G}_2(x)}{1+(x/2)^2}, \tag{99}$$

where

$$\tilde{G}_2(x) = \int_0^x (x-g)^{\gamma_2} g^{\gamma_3+(1/k_1)} e^{\nu g} dg \tag{100}$$

for all $x \ge 0$. From the sharp bounds established in Proposition 8 of [30], we can pinpoint a constant $K_1 > 0$ (depending on $\gamma_2, \gamma_3, k_1, \nu$) with

$$\check{G}_2(x) \le K_1 x^{\gamma_3+(1/k_1)} e^{\nu x} \tag{101}$$

for all $x \ge 1$, under the conditions (87). As a result, we get that

$$G_2(x) \le K_1 \frac{x^{\gamma_3+(1/k_1)}}{1+(x/2)^2} e^{\nu x} \tag{102}$$

provided that $x \ge 1$.

At last, gathering the bounds (97), (98), and (102), we deduce the awaited bounds (93) from the splitting (95). □

We turn to the bounds for the map $\mathscr{B}(\tau, m)$. We identify two alternatives.

(i) Assume that $\tau \in S_{d_1} \cup D_\rho$ is chosen such that

$$\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}} > 1. \tag{103}$$

Owing to Lemma 4 and the first constraint of (87), we get from the upper bounds (91) some constant $C_{1.1} > 0$ with

$$\mathcal{B}(\tau, m) \leq \frac{M_{\gamma_1} \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}} |\tau|^{k_1} |\epsilon|^{k_1(\gamma_2+\gamma_3+1)} K_G \cdot \frac{\left(||^{k_1}\right)^{\gamma_3+(1/k_1)}}{1+|\tau/\epsilon|^{2k_1}} \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1+|m|)^{-\mu} e^{-\beta|m|} \leq C_{1.1} M_{\gamma_1} |\epsilon|^{k_1(\gamma_2+1)} \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}} \cdot \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1+|m|)^{-\mu} e^{-\beta|m|}, \quad (104)$$

for all $\tau \in S_{d_1} \cup D_\rho$ chosen under (103).

(ii) Suppose that $\tau \in S_{d_1} \cup D_\rho$ fulfills

$$0 \leq \frac{|\tau|^{k_1}}{|\epsilon|^{k_1}} \leq 1. \quad (105)$$

Based on (91), we arrive at some constant $C_{1.2} > 0$ such that

$$\mathcal{B}(\tau, m) \leq \frac{M_{\gamma_1} \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}}{(1+|\tau|)^{\gamma_1}} |\tau|^{k_1} \int_0^{|\tau|^{k_1}/|\epsilon|^{k_1}} \left(\frac{|\tau|^{k_1}}{|\epsilon|^{k_1}} - g\right)^{\gamma_2} \cdot g^{\gamma_3+(1/k_1)} \frac{1}{1+g^2} dg \times \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) |\epsilon|^{k_1(\gamma_2+\gamma_3+1)} (1+|m|)^{-\mu} e^{-\beta|m|} \leq \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}} \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1+|m|)^{-\mu} e^{-\beta|m|} \times \left[C_{1.2} |\tau|^{k_1-1} |\epsilon|^{k_1(\gamma_2+1)} |\epsilon|^{1+k_1\gamma_3} M_{\gamma_1} \left(1+\left|\frac{\tau}{\epsilon}\right|^{2k_1}\right)\right] \leq \left[C_{1.2} M_{\gamma_1} \epsilon_0^{k_1(\gamma_3+1)} 2\right] |\epsilon|^{k_1(\gamma_2+1)} \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \cdot \frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}} \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1+|m|)^{-\mu} e^{-\beta|m|}, \quad (106)$$

whenever $\tau \in S_{d_1} \cup D_\rho$ is restricted to (105).

Eventually, the combination of the above bounds (104) and (106) yields the expected result (88).

**Proposition 10.** *Let $Q(X), R(X) \in \mathbb{C}[X]$ be polynomials and $\mu > 0$ be a real number subjected to the constraints*

$$\deg(R) \geq \deg(Q), R\left(\sqrt{-1}m\right) \neq 0, \mu > \deg(Q) + 1, \quad (107)$$

*for all $m \in \mathbb{R}$. Then, a constant $C_2 > 0$ (depending on Q,R, and $\mu$) can be selected such that*

$$\left\|\frac{1}{R\left(\sqrt{-1}m\right)} \int_{-\infty}^{+\infty} f(m-m_1) Q\left(\sqrt{-1}m_1\right) g(\tau, m_1) dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq C_2 \|f(m)\|_{(\beta,\mu)} \|g(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \quad (108)$$

*holds provided that $f \in E_{(\beta,\mu)}$ and $g \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\varepsilon)}$.*

*Proof.* The proof mirrors the one of Proposition 10 in our recent work [34]. Indeed, let us choose $g$ in $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. The very definition of the norms displayed in Definitions 3 and 4 allows the bounds

$$|g(\tau, m_1)| \leq \|g\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \left|\frac{\tau}{\epsilon}\right| \frac{1}{1+|\tau/\epsilon|^{2k_1}} \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1+|m|)^{-\mu} e^{-\beta|m_1|}, \quad (109)$$

provided that $\tau_1 \in S_{d_1} \cup D_\rho$ and $m_1 \in \mathbb{R}$ together with

$$|f(m)| \leq \|f(m)\|_{(\beta,\mu)} (1+|m|)^{-\mu} e^{-\beta|m|}, \quad (110)$$

for all $m \in \mathbb{R}$. These two bounds (109) and (110) yield the next estimates

$$|\mathcal{C}(\tau, m)| := \left|\frac{1}{R\left(\sqrt{-1}m\right)} \int_{-\infty}^{+\infty} f(m-m_1) Q\left(\sqrt{-1}m_1\right) g(\tau, m_1) dm_1\right| \leq \|f(m)\|_{(\beta,\mu)} \|g(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \left|\frac{\tau}{\epsilon}\right| \frac{1}{1+|\tau/\epsilon|^{2k_1}} \cdot \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1+|m|)^{-\mu} e^{-\beta|m|} C_{2.1}, \quad (111)$$

where

$$C_{2.1} = (1+|m|)^{\mu} e^{\beta|m|} \frac{1}{\left|R\left(\sqrt{-1}m\right)\right|} \int_{-\infty}^{+\infty} \frac{e^{-\beta|m-m_1|}}{(1+|m-m_1|)^{\mu}} \cdot \frac{\left|Q\left(\sqrt{-1}m_1\right)\right|}{(1+|m_1|)^{\mu}} e^{-\beta|m_1|} dm_1. \quad (112)$$

According to the triangular inequality, we observe that

$$|m| \le |m - m_1| + |m_1|, \tag{113}$$

for all real numbers $m, m_1 \in \mathbb{R}$ and by the construction of the polynomials $R, Q$ asked to fulfill (107), two constants $\mathfrak{Q}, \mathfrak{R} > 0$ can be pinpointed such that

$$\left|Q\left(\sqrt{-1}m_1\right)\right| \le \mathfrak{Q}(1 + |m_1|)^{\deg (Q)}, \left|R\left(\sqrt{-1}m\right)\right| \ge \mathfrak{R}(1 + |m|)^{\deg (R)}, \tag{114}$$

whenever $m, m_1 \in \mathbb{R}$. Thereby, the next upper bounds

$$C_{2.1} \le \frac{\mathfrak{Q}}{\mathfrak{R}} \sup_{m\in\mathbb{R}} (1 + |m|)^{\mu - \deg (R)} \cdot \int_{-\infty}^{+\infty} \frac{1}{(1 + |m - m_1|)^{\mu} (1 + |m_1|)^{\mu - \deg (Q)}} dm_1, \tag{115}$$

are reached whose right hand side is a finite quantity under the restrictions (107), owing to Lemma 2.2 from [29] or Lemma 4 of [35].

Eventually, gathering (111) and (115) yields the foretold bounds (108). □

**Proposition 11.** *Let $k_1 \ge 1$ be an integer. Let $Q_1(X), Q_2(X)$, and $R(X)$ be polynomials with complex coefficients such that*

$$\deg (R) \ge \deg (Q_1), \deg (R) \ge \deg (Q_2), R\left(\sqrt{-1}m\right) \ne 0, \tag{116}$$

*for all $m \in \mathbb{R}$. We require the positive real number $\mu > 0$ to satisfy*

$$\mu > \max (\deg (Q_1) + 1, \deg (Q_2) + 1). \tag{117}$$

*Let $m \mapsto b(m)$ be a continuous function on $\mathbb{R}$ such that*

$$|b(m)| \le \frac{1}{\left|R\left(\sqrt{-1}m\right)\right|}, \tag{118}$$

*for all $m \in \mathbb{R}$. Then, one can find a constant $C_3 > 0$ (relying on $Q_1$, $Q_2$, $R$, $\mu$, $k_1$, and $\nu$) such that*

$$\begin{aligned} &\left\| b(m)\tau^{k_1} \int_0^{\tau^{k_1}} \int_{-\infty}^{+\infty} Q_1\left(\sqrt{-1}(m - m_1)\right) f\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) \right. \\ &\quad \left. \times Q_2\left(\sqrt{-1}m_1\right) g\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\le C_3 \|f(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \|g(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \end{aligned} \tag{119}$$

*for all $f, g \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*

*Proof.* Take $f, g$ in the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. According to the definition of the norm, the next two bounds

$$\begin{aligned} |f(\tau, m)| &\le \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \left|\frac{\tau}{\varepsilon}\right| \frac{1}{1 + |\tau/\epsilon|^{2k_1}} \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1 + |m|)^{-\mu} e^{-\beta|m|}, \\ |g(\tau, m)| &\le \|g\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \left|\frac{\tau}{\varepsilon}\right| \frac{1}{1 + |\tau/\epsilon|^{2k_1}} \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times (1 + |m|)^{-\mu} e^{-\beta|m|}, \end{aligned} \tag{120}$$

hold provided that $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$. These bounds together with the assumption (118) prompt

$$\begin{aligned} \mathscr{D}(\tau, m) &:= \left| b(m)\tau^{k_1} \int_0^{\tau^{k_1}} \int_{-\infty}^{+\infty} Q_1\left(\sqrt{-1}(m - m_1)\right) \cdot f\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) \right. \\ &\quad \left. \times Q_2\left(\sqrt{-1}m_1\right) g\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1 \right| \\ &\le \frac{1}{\left|R\left(\sqrt{-1}m\right)\right|} \int_{-\infty}^{+\infty} \left|Q_1\left(\sqrt{-1}(m - m_1)\right)\right| \cdot \left|Q_2\left(\sqrt{-1}m_1\right)\right| (1 + |m - m_1|)^{-\mu} e^{-\beta|m - m_1|} \\ &\quad \times (1 + |m_1|)^{-\mu} e^{-\beta|m_1|} dm_1 \|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \|g\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad \times |\tau|^{k_1} \int_0^{|\tau|^{k_1}} \frac{\left(|\tau|^{k_1} - h\right)^{1/k_1}}{|\epsilon|} \frac{1}{1 + \left(\left(|\tau|^{k_1} - h\right)^2 / |\epsilon|^{2k_1}\right)} \\ &\quad \cdot \frac{h^{1/k_1}}{|\epsilon|} \frac{1}{1 + \left(h^2/|\epsilon|^{2k_1}\right)} \frac{1}{\left(|\tau|^{k_1} - h\right)h} dh \\ &\quad \times \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right), \end{aligned} \tag{121}$$

for all $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$.

By construction, we check that some positive constants $\mathfrak{Q}_1$,$\mathfrak{Q}_2$, and $\mathfrak{R}$ can be picked out in a way that

$$\begin{aligned} \left|Q_1\left(\sqrt{-1}(m - m_1)\right)\right| &\le \mathfrak{Q}_1(1 + |m - m_1|)^{\deg (Q_1)}, \left|Q_2\left(\sqrt{-1}m_1\right)\right| \\ &\le \mathfrak{Q}_2(1 + |m_1|)^{\deg (Q_2)}, \left|R\left(\sqrt{-1}m\right)\right| \\ &\ge \mathfrak{R}(1 + |m|)^{\deg (R)}, \end{aligned} \tag{122}$$

for all $m, m_1 \in \mathbb{R}$. As a result and keeping in mind the inequality (113), we deduce the next bounds for the first piece of the right handside of (121), namely,

$$\frac{1}{\left|R\left(\sqrt{-1}m\right)\right|}\int_{-\infty}^{+\infty}\left|Q_1\left(\sqrt{-1}(m-m_1)\right)\right|\left|Q_2\left(\sqrt{-1}m_1\right)\right| \cdot (1+|m-m_1|)^{-\mu}e^{-\beta|m-m_1|} \times (1+|m_1|)^{-\mu}e^{-\beta|m_1|}dm_1 \leq \frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}}\mathcal{D}(1+|m|)^{-\mu}e^{-\beta|m|}, \tag{123}$$

where

$$\mathcal{D} := \sup_{m\in\mathbb{R}}(1+|m|)^{\deg(R)} \cdot \int_{-\infty}^{+\infty}\frac{1}{(1+|m-m_1|)^{\mu-\deg(Q_1)}(1+|m_1|)^{\mu-\deg(Q_2)}}dm_1, \tag{124}$$

is a finite quantity under conditions (116) and (117), as explained in Lemma 6.2 from [29] or Lemma 4 of [35]. Besides, according to Lemma 7 of our recent work [36], there exists a constant $K_{k_1}$ (relying on $k_1$) such that

$$|\tau|^{k_1}\int_0^{|\tau|^{k_1}}\frac{\left(|\tau|^{k_1}-h\right)^{1/k_1}/|\epsilon|}{1+\left(\left(|\tau|^{k_1}-h\right)^2/|\epsilon|^{2k_1}\right)} \cdot \frac{h^{1/k_1}/|\epsilon|}{1+\left(h^2/|\epsilon|^{2k_1}\right)}\frac{1}{\left(|\tau|^{k_1}-h\right)h}dh \leq K_{k_1}\frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}}, \tag{125}$$

for all $\tau \in S_{d_1} \cup D_\rho$, all $\varepsilon \in D_{\varepsilon_0} \setminus \{0\}$.

Counting up the above two bounds (123) and (125), it results from (121) that

$$\mathcal{D}(\tau, m) \leq \frac{\mathfrak{Q}_1\mathfrak{Q}_2}{\mathfrak{R}}\mathcal{D}K_{k_1}\|f\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\|g\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \cdot \frac{|\tau/\epsilon|}{1+|\tau/\epsilon|^{2k_1}}(1+|m|)^{-\mu}e^{-\beta|m|} \times \exp\left(\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right), \tag{126}$$

whenever $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$. The estimates (119) follow. □

## 5. Solving the First Convolution Equation (83)

In this section, we uniquely solve the auxiliary convolution Equation (83) stated in Subsection 3.2 within the Banach spaces displayed in Definition 4. Our approach consists in rearranging (83) into a fixed point equation (disclosed later on in (173)). In the first stage, we ask to perform a division by the next parameter depending polynomial

$$P_m(\tau) = Q\left(\sqrt{-1}m\right) - R_D\left(\sqrt{-1}m\right)k_1^{\delta_D}\tau^{k_1\delta_D}, \tag{127}$$

provided that $\tau \in S_{d_1} \cup D_\rho$. Decisive lower bounds concerning $P_m$ are displayed in the next lemma.

**Lemma 12.** *For a convenient choice of the inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector $S_{Q,R_D}$ (introduced in (44)), unbounded sectors $S_{d_1}$ centered at 0 with bisecting direction $d_1 \in \mathbb{R}$ and a small radius $\rho > 0$ can be distinguished in a way that the next lower estimates*

$$|P_m(\tau)| \geq C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}\left|R_D\left(\sqrt{-1}m\right)\right|(1+|\tau|)^{k_1\delta_D-1}, \tag{128}$$

*hold for some well-chosen constant $C_P > 0$, provided that $\tau \in S_{d_1} \cup D_\rho$, for all $m \in \mathbb{R}$.*

*Proof.* Owing to the fact that the complex roots $q_l(m)$, $0 \leq l \leq k_1\delta_D - 1$ of $\tau \mapsto P_m(\tau)$ can be explicitely computed, we factorize the polynomial as follows:

$$P_m(\tau) = -R_D\left(\sqrt{-1}m\right)k_1^{\delta_D}\Pi_{l=0}^{k_1\delta_D-1}(\tau - q_l(m)), \tag{129}$$

with

$$q_l(m) = \left(\frac{\left|Q\left(\sqrt{-1}m\right)\right|}{\left|R_D\left(\sqrt{-1}m\right)\right|k_1^{\delta_D}}\right)^{1/k_1\delta_D}\exp \cdot \left(\sqrt{-1}\left(\arg\left(\frac{Q\left(\sqrt{-1}m\right)}{R_D\left(\sqrt{-1}m\right)k_1^{\delta_D}}\right)\frac{1}{k_1\delta_D} + \frac{2\pi l}{k_1\delta_D}\right)\right), \tag{130}$$

for all $0 \leq l \leq k_1\delta_D - 1$, for any $\tau \in \mathbb{C}$ and $m \in \mathbb{R}$.

We pinpoint an unbounded sector $S_{d_1}$ centered at 0, a small disc $D_\rho$ and we position the sector $S_{Q,R_D}$ given in (44) in a way that the next two properties hold:

(1) A constant $M_1 > 0$ can be found such that

$$|\tau - q_l(m)| \geq M_1(1+|\tau|), \tag{131}$$

for all $0 \leq l \leq k_1\delta_D - 1$, all $m \in \mathbb{R}$, whenever $\tau \in S_{d_1} \cup D_\rho$.

(2) There exists a constant $M_2 > 0$ with

$$\left|\tau - q_{l_0}(m)\right| \geq M_2 \left|q_{l_0}(m)\right|, \quad (132)$$

for some $0 \leq l_0 \leq \delta_D k_1 - 1$, all $m \in \mathbb{R}$, all $\tau \in S_{d_1} \cup D_\rho$.

We now explain how the above two bounds can be established.

(i) We deem the first inequality (131) in observing that under hypothesis (45); the roots $q_l(m)$ are bounded from below and obey $|q_l(m)| \geq 2\rho$ for all $m \in \mathbb{R}$, all $0 \leq l \leq \delta_D k_1 - 1$ for a suitable choice of the radii $r_{Q,R_D}, \rho > 0$. Furthermore, for all $m \in \mathbb{R}$, all $0 \leq l \leq \delta_D k_1 - 1$; these roots are penned inside an union $\mathcal{Q}$ of unbounded sectors centered at 0 that do not cover a full neighborhood of 0 in $\mathbb{C}^*$ whenever the aperture $\eta_{Q,R_D} > 0$ of $S_{Q,R_D}$ is taken small enough. Hence, a sector $S_{d_1}$ may be chosen such that

$$S_{d_1} \cap \mathcal{Q} = \varnothing. \quad (133)$$

Such a sector satisfies in particular that for all $0 \leq l \leq \delta_D k_1 - 1$, the quotients $q_l(m)/\tau$ lay outside some small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_{d_1}$, all $m \in \mathbb{R}$. Eventually, (131) follows.

The sector $S_{d_1}$ and disc $D_\rho$ are selected as above. The second lower bound (132) ensues from the fact that for any fixed $0 \leq l_0 \leq \delta_D k_1 - 1$, the quotient $\tau/q_{l_0}(m)$ stays apart a small disc centered at 1 in $\mathbb{C}$ for all $\tau \in S_{d_1} \cup D_\rho$, all $m \in \mathbb{R}$.

Departing from factorization (129) and paying regard to the two lower bounds (131) and (132) reached overhead, we arrive at

$$\begin{aligned} |P_m(\tau)| &\geq M_1^{k_1\delta_D - 1} M_2 \left|R_D\left(\sqrt{-1}m\right)\right| k_1^{\delta_D} \cdot \left(\frac{\left|Q\left(\sqrt{-1}m\right)\right|}{\left|R_D\left(\sqrt{-1}m\right)\right| k_1^{\delta_D}}\right)^{1/k_1\delta_D} (1+|\tau|)^{k_1\delta_D - 1} \\ &\geq C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D} \left|R_D\left(\sqrt{-1}m\right)\right| (1+|\tau|)^{k_1\delta_D - 1}, \end{aligned} \quad (134)$$

as long as $\tau \in S_{d_1} \cup D_\rho$, for all $m \in \mathbb{R}$. □

We introduce the next nonlinear map

$$\begin{aligned} \mathcal{H}_\epsilon(\omega(\tau,m)) &:= \left(\sum_{q=1}^{\delta_D - 1} a_{q,\delta_D} \left[\frac{\tau^{k_1}}{P_m(\tau)\Gamma\left(d_{D,q}/k_1\right)} \cdot \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\left(d_{D,q}/k_1\right)-1} k_1^q s^q \omega\left(s^{1/k_1}, m\right) \frac{ds}{s} \right.\right. \\ &+ \sum_{1 \leq p \leq q-1} A_{q,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma\left(d_{D,q} + k_1(q-p)/k_1\right)} \\ &\left.\cdot \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\left(d_{D,q} + k_1(q-p)/k_1\right)-1} k_1^p s^p \omega\left(s^{1/k_1}, m\right) \frac{ds}{s}\right] \\ &\left.\times R_D\left(\sqrt{-1}m\right)\right) + \left[\sum_{1 \leq p \leq \delta_D - 1} A_{\delta_D,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma(\delta_D - p)} \cdot \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\delta_D - p - 1} k_1^p s^p \omega\left(s^{1/k_1}, m\right) \frac{ds}{s}\right] \times R_D\left(\sqrt{-1}m\right) \\ &+ \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} \left[\sum_{q=1}^{\delta_l} a_{q,\delta_l} \left[\frac{\tau^{k_1}}{P_m(\tau)\Gamma\left(d_{l,q}/k_1\right)} \cdot \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\left(d_{l,q}/k_1\right)-1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m - m_1, \epsilon) k_1^q s^q \right.\right. \\ &\times R_l\left(\sqrt{-1}m_1\right) \omega\left(s^{1/k_1}, m_1\right) \frac{ds}{s} dm_1 \\ &+ \sum_{1 \leq p \leq q-1} A_{q,p} \frac{\tau^{k_1}}{P_m(\tau)\Gamma\left(d_{l,q} + k_1(q-p)/k_1\right)} \cdot \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\left(d_{l,q} + k_1(q-p)/k_1\right)-1} \\ &\times \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} A_l(m - m_1, \epsilon) k_1^p s^p R_l\left(\sqrt{-1}m_1\right) \omega\left(s^{1/k_1}, m_1\right) \\ &\left.\left.\cdot \frac{ds}{s} dm_1\right]\right] + \frac{\mathcal{F}_2(\tau, m, \epsilon)}{P_m(\tau)} + \frac{1}{(2\pi)^{1/2} P_m(\tau)} \\ &\cdot \int_{-\infty}^{+\infty} C_1(m - m_1, \epsilon) \omega(\tau, m_1) dm_1 + c_{Q_1 Q_2} \frac{1}{(2\pi)^{1/2} P_m(\tau)} \\ &\cdot \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} Q_1\left(\sqrt{-1}(m - m_1)\right) \omega\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) \\ &\times Q_2\left(\sqrt{-1}m_1\right) \omega\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1. \end{aligned} \quad (135)$$

In the next proposition, we establish that $\mathcal{H}_\epsilon$ represents a shrinking map on some suitable ball of the Banach space mentioned in Definition 4.

**Proposition 13.** *Let us select a well-chosen inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector $S_{Q,R_D}$ jointly with an unbounded sector $S_{d_1}$ and radius $\rho > 0$ that heed the requirements of Lemma 12 and obey the additional condition*

$$-1 \notin S_{d_1} \cup D_\rho. \quad (136)$$

*Then, one can single out a radius $\epsilon_0 > 0$ small enough, constants $C_{1,\epsilon_0} > 0$ and $c_{Q_1,Q_2} \in \mathbb{C}^*$ close enough to 0 and a fitting radius $\varpi_2 > 0$ in a way that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the map $\mathcal{H}_\epsilon$ enjoys the next two features*

*(i) The inclusion*

$$\mathcal{H}_\epsilon\left(\bar{B}_{\varpi_2}\right) \subset \bar{B}_{\varpi_2}, \quad (137)$$

*holds, where we denote $\bar{B}_{\varpi_2}$, the closed ball of radius $\varpi_2 > 0$ centered at 0 in the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*

*(ii) The 1/2 – Lipschitz condition*

$$\begin{aligned}\|\mathscr{H}_\epsilon(\omega_1) - \mathscr{H}_\epsilon(\omega_2)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ \leq \frac{1}{2}\|\omega_1 - \omega_2\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},\end{aligned} \tag{138}$$

*occurs for all $\omega_1, \omega_2 \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*

*Proof.* We take aim at the first item stating the inclusion (137). We prescribe some real number $\varpi_2 > 0$ and take $\omega(\tau, m)$ in $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$, for given $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, such that

$$\|\omega\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \varpi_2. \tag{139}$$

We provide explicit bounds for each term of the map $\mathscr{H}_\epsilon$ applied to $\omega$.

According to Proposition 8 and Lemma 12, we observe that

$$\begin{aligned}&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1}s^q R_D\left(\sqrt{-1}m\right)\omega\left(s^{1/k_1},m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad\leq \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-q)k_1}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},\end{aligned} \tag{140}$$

for $1 \leq q \leq \delta_D - 1$ along with

$$\begin{aligned}&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1}s^q R_D\left(\sqrt{-1}m\right)\omega\left(s^{1/k_1},m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad\leq \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},\end{aligned} \tag{141}$$

for $1 \leq p \leq q-1$ with $1 \leq q \leq \delta_D - 1$ and

$$\begin{aligned}&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\delta_D-p-1}s^q R_D\left(\sqrt{-1}m\right)\omega\left(s^{1/k_1},m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad\leq \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},\end{aligned} \tag{142}$$

as long as $1 \leq p \leq \delta_D - 1$. In order to handle the next piece, under the constraint (136), we can recast

$$\begin{aligned}&\mathscr{E}_1(\tau,m,\epsilon) \\ &:= \frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^q \\ &\quad\times R_l\left(\sqrt{-1}m_1\right)\omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}dm_1 \\ &= \frac{R_D\left(\sqrt{-1}m\right)(1+\tau)^{k_1\delta_D-1}}{P_m(\tau)} \times \frac{1}{R_D\left(\sqrt{-1}m\right)} \\ &\quad\cdot\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)R_l\left(\sqrt{-1}m_1\right) \\ &\quad\times\left[\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}s^q\omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}\right]dm_1,\end{aligned} \tag{143}$$

for all $\tau \in S_{d_1} \cup D_\rho$, $m \in \mathbb{R}$ with $1 \leq l \leq D-1$ and $1 \leq q \leq \delta_l$. Based on Lemma 12, we check that

$$\left|\frac{R_D\left(\sqrt{-1}m\right)(1+\tau)^{k_1\delta_D-1}}{P_m(\tau)}\right| \leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}, \tag{144}$$

provided that $\tau \in S_{d_1} \cup D_\rho$, $m \in \mathbb{R}$. Owing to assumptions (42) and (48), Proposition 10 together with (144) yields

$$\begin{aligned}&\|\mathscr{E}_1(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad\leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\|A_l(m,\epsilon)\|_{(\beta,\mu)} \\ &\quad\times\left\|\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1}s^q\omega\left(s^{1/k_1},m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned} \tag{145}$$

Besides, a constant $M_{k_1,\delta_D} > 0$ can be picked up such that

$$\left|\frac{1}{(1+\tau)^{k_1\delta_D-1}}\right| \leq \frac{M_{k_1,\delta_D}}{(1+|\tau|)^{k_1\delta_D-1}}, \tag{146}$$

for all $\tau \in S_{d_1} \cup D_\rho$, assuming condition (136). Condition (40) together with (146) enables us to apply Proposition 8 and prompt

$$\begin{aligned}&\left\|\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1}s^q\omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad\leq C_1 M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned} \tag{147}$$

Eventually, bearing in mind (55), we deduce from (145) complemented by (147) that

$$\|\mathscr{E}_1(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \mathrm{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}} \|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \quad (148)$$

The ensuing block is remodeled as

$$\begin{aligned}\mathscr{E}_2(\tau,m,\epsilon) &:= \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} \left(\tau^{k_1}-s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1} \\ &\quad \times \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) s^p R_l\left(\sqrt{-1}m_1\right) \\ &\quad \cdot \omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}\,dm_1 \\ &= \frac{R_D\left(\sqrt{-1}m\right)(1+\tau)^{k_1\delta_D-1}}{P_m(\tau)} \times \frac{1}{R_D\left(\sqrt{-1}m\right)} \\ &\quad \cdot \int_{-\infty}^{+\infty} A_l(m-m_1,\epsilon) R_l\left(\sqrt{-1}m_1\right) \\ &\quad \times \left[\frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}} \int_0^{\tau^{k_1}} \left(\tau^{k_1}-s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1} \right. \\ &\quad \left. \cdot s^p \omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}\right] dm_1,\end{aligned} \quad (149)$$

for all $\tau \in S_{d_1} \cup D_\rho$, $m \in \mathbb{R}$ with $1 \leq l \leq D-1$, $1 \leq q \leq \delta_l$ and $1 \leq p \leq q-1$, under (115).

Assumptions (42) and (48) and the upper bounds (144) warrant the application of Proposition 10 which triggers

$$\begin{aligned}\|\mathscr{E}_1(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} &\leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \|A_l(m,\varepsilon)\|_{(\beta,\mu)} \times \left\| \frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}} \right. \\ &\quad \left. \cdot \int_0^{\tau^{k_1}} \left(\tau^{k_1}-s\right)^{d_{l,q}+k_1(q-p)/k_1-1} s^p \omega\left(s^{1/k_1},m\right)\frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned} \quad (150)$$

Condition (40) coupled with (146) grants the use of Proposition 8 and beget

$$\begin{aligned}&\left\| \frac{\tau^{k_1}}{(1+\tau)^{k_1\delta_D-1}} \int_0^{\tau^{k_1}} \left(\tau^{k_1}-s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1} s^p \omega\left(s^{1/k_1},m_1\right)\frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad \leq C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}+k_1(q-p)} \|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned} \quad (151)$$

At last, not forgetting (55), we deduce from the joint bounds (150) and (151) that

$$\begin{aligned}\|\mathscr{E}_2(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} &\leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \mathrm{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}+k_1(q-p)} \\ &\quad \cdot \|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned} \quad (152)$$

We control now the piece $\mathscr{F}_2(\tau,m,\epsilon)/P_m(\tau)$. In accordance with Lemma 12, we notice that

$$\left|\frac{1}{P_m(\tau)}\right| \leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|, \quad (153)$$

provided that $\tau \in S_{d_1} \cup D_\rho$ and $m \in \mathbb{R}$, whose right handside is a finite quantity since $R_D(\sqrt{-1}m) \neq 0$ holds from (45) for all $m \in \mathbb{R}$. Besides, owing to the definition of $\mathscr{F}_2$ given in Subsection 2.2 and bounds (49), we deduce

$$|\mathscr{F}_2(\tau,m,\epsilon)| \leq \sum_{j_2\in J_2} \mathrm{F}_{2,j_2,\epsilon_0}(1+|m|)^{-\mu} e^{-\beta|m|} |\tau|^{j_2}, \quad (154)$$

for all $\tau \in \mathbb{C}$, $m \in \mathbb{R}$. The combination of the bounds (153) and (154) grants

$$\begin{aligned}&\|\mathscr{F}_2(\tau,m,\epsilon)/P_m(\tau)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\quad \leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right| \\ &\quad \times \sup_{\tau\in S_{d_1}\cup D_\rho, m\in\mathbb{R}} (1+|m|)^{\mu} e^{\beta|m|} \left|\frac{\epsilon}{\tau}\right| \left(1+\left|\frac{\tau}{\epsilon}\right|^{2k_1}\right) \\ &\quad \cdot \exp\left(-\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right) \times \left(\sum_{j_2\in J_2} \mathrm{F}_{2,j_2,\epsilon_0}(1+|m|)^{-\mu} e^{-\beta|m|} |\tau|^{j_2}\right) \\ &\quad \leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right| \sup_{\tau\in S_{d_1}\cup D_\rho} \\ &\quad \cdot \exp\left(-\nu\left|\frac{\tau}{\epsilon}\right|^{k_1}\right)\left(1+\left|\frac{\tau}{\epsilon}\right|^{2k_1}\right) \times \left(\sum_{j_2\in J_2} \mathrm{F}_{2,j_2,\epsilon_0} |\epsilon|^{j_2} \left|\frac{\tau}{\epsilon}\right|^{j_2-1}\right) \\ &\quad \leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right| \epsilon_0 \\ &\quad \times \sup_{x\geq 0} e^{-\nu x^{k_1}} \left(1+x^{2k_1}\right) \sum_{j_2\in J_2} \mathrm{F}_{2,j_2,\epsilon_0} \epsilon_0^{j_2-1} x^{j_2-1},\end{aligned} \quad (155)$$

which represents a finite quantity bearing in mind that $J_2 \subset \mathbb{N}^*$ contains only positive integers.

We address the ensuing linear part of $\mathscr{H}_\epsilon$. Paying regard to (153) and the bounds (55), Proposition 10 prompts

$$\begin{aligned}&\left\|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}C_1(m-m_1,\epsilon)\omega(\tau,m_1)dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\leq\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|\\&\qquad\cdot C_2\|C_1(m,\epsilon)\|_{(\beta,\mu)}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\leq\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2\mathbf{C}_{1,\epsilon_0}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned}\tag{156}$$

At last, we manage the nonlinear tail piece of $\mathscr{H}_\epsilon$. We first factorize

$$\frac{1}{P_m(\tau)}=\frac{1}{R_D\left(\sqrt{-1}m\right)}\mathscr{G}(\tau,m),\tag{157}$$

where

$$|\mathscr{G}(\tau,m)|\leq\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}},\tag{158}$$

for all $\tau\in S_{d_1}\cup D_\rho$ and $m\in\mathbb{R}$, according to (128). This latter decomposition together with the assumption (43) enable the application of Proposition 11 which yields

$$\begin{aligned}&\left\|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}Q_1\left(\sqrt{-1}(m-m_1)\right)\right.\\&\qquad\cdot\omega\left(\left(\tau^{k_1}-s\right)^{1/k_1},m-m_1\right)\\&\qquad\left.\times Q_2\left(\sqrt{-1}m_1\right)\omega\left(s^{1/k_1},m_1\right)\frac{1}{\left(\tau^{k_1}-s\right)s}dsdm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\leq\frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\|\omega(\tau,m)\|^2_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned}\tag{159}$$

We select $\epsilon_0>0$, $\mathrm{C}_{1,\epsilon_0}>0$, and $c_{Q_1,Q_2}\in\mathbb{C}^*$ close enough to 0 and take suitably $\varpi_2>0$ in a proper way that the next inequality

$$\begin{aligned}&\left(\sum_{q=1}^{\delta_D-1}\left|a_{q,\delta_D}\right|\left[\frac{1}{\Gamma\left(d_{D,q}/k_1\right)}k_1^q\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\epsilon_0^{(\delta_D-q)k_1}\varpi_2\right.\right.\\&\quad+\sum_{1\leq p\leq q-1}\left|A_{q,p}\right|\frac{1}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)}\\&\quad\left.\left.\cdot k_1^p\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\epsilon_0^{(\delta_D-p)k_1}\varpi_2\right]\right)\\&\quad+\left[\sum_{1\leq p\leq\delta_D-1}\left|A_{\delta_D,p}\right|\frac{k_1^p}{\Gamma(\delta_D-p)}\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\epsilon_0^{(\delta_D-p)k_1}\varpi_2\right]\\&\quad+\sum_{l=1}^{D-1}\epsilon_0^{\Delta_l-d_l}\left[\sum_{q=1}^{\delta_l}\left|a_{q,\delta_l}\right|\left[\frac{k_1^q}{\Gamma\left(d_{l,q}/k_1\right)}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\right.\right.\\&\quad\cdot C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}}\varpi_2+\sum_{1\leq p\leq q-1}\left|A_{q,p}\right|\\&\quad\cdot\frac{k_1^p}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)}\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\\&\quad\left.\left.\cdot C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}+k_1(q-p)}\varpi_2\right]\right]+\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\\&\quad\cdot\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|\epsilon_0\times\sup_{x\geq0}e^{-\nu x^{k_1}}\left(1+x^{2k_1}\right)\\&\quad\cdot\sum_{j_2\in J_2}\mathrm{F}_{2,j_2,\epsilon_0}\epsilon_0^{j_2-1}x^{j_2-1}+\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\\&\quad\cdot\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2\mathbf{C}_{1,\epsilon_0}\varpi_2\\&\quad+\left|c_{Q_1,Q_2}\right|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\varpi_2^2\leq\varpi_2,\end{aligned}\tag{160}$$

holds. Observe that the first six blocks of the left handside of (160) can be made small since they contain positive powers of $\epsilon_0$, owing in particular to the constraint (41) imposed on (37) and its last two terms can be dwindled provided that the positive constants $\mathbf{C}_{1,\epsilon_0}$ and $c_{Q_1,Q_2}$ are chosen nearby the origin.

Eventually, the collection of all the bounds overhead (140)–(142), (148), (152), (155), (156), and (159) restricted by (160) gives rise to inclusion (137).

We mind the second item addressing the 1/2 – Lipschitz feature. Take $\omega_1,\omega_2$ inside the ball $\bar{B}_{\varpi_2}$ of the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ whose radius $\varpi_2$ has been prescribed in the first item discussed above. We display norm estimates for each block of the difference $\mathscr{H}_\epsilon(\omega_1)-\mathscr{H}_\epsilon(\omega_2)$. Based on the bounds reached formerly in the proof of the first item, we check the next list of six estimates. Namely,

$$\begin{aligned}&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\left(d_{D,q}/k_1\right)-1}s^qR_D\left(\sqrt{-1}m\right)\right.\\&\qquad\left.\cdot\left(\omega_1\left(s^{1/k_1},m\right)-\omega_2\left(s^{1/k_1},m\right)\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\leq\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-q)k_1}\|\omega_1(\tau,m)-\omega_2(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\end{aligned}\tag{161}$$

for $1 \le q \le \delta_D - 1$ along with

$$\left\| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\left(d_{D,q}+k_1(q-p)/k_1\right)-1} s^q R_D\left(\sqrt{-1}m\right) \times \left(\omega_1\left(s^{1/k_1}, m\right) - \omega_2\left(s^{1/k_1}, m\right)\right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} |\epsilon|^{(\delta_D - p)k_1} \|\omega_1(\tau, m) - \omega_2(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{162}$$

for $1 \le p \le q - 1$ with $1 \le q \le \delta_D - 1$ and

$$\left\| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\delta_D - p - 1} s^p R_D\left(\sqrt{-1}m\right) \cdot \left(\omega_1\left(s^{1/k_1}, m\right) - \omega_2\left(s^{1/k_1}, m\right)\right) \frac{ds}{s} \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} |\epsilon|^{(\delta_D - p)k_1} \|\omega_1(\tau, m) - \omega_2(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{163}$$

as long as $1 \le p \le \delta_D - 1$. Furthermore,

$$\left\| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{(d_{l,q}/k_1)-1} \int_{-\infty}^{+\infty} A_l(m - m_1, \epsilon) s^q R_l \cdot \left(\sqrt{-1}m_1\right) \times \left(\omega_1\left(s^{1/k_1}, m_1\right) - \omega_2\left(s^{1/k_1}, m_1\right)\right) \frac{ds}{s} dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \mathbf{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}} \cdot \|\omega_1(\tau, m) - \omega_2(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{164}$$

holds for $1 \le l \le D - 1$ and $1 \le q \le \delta_l$ together with

$$\left\| \frac{\tau^{k_1}}{P_m(\tau)} \int_0^{\tau^{k_1}} \left(\tau^{k_1} - s\right)^{\left(d_{l,q}+k_1(q-p)/k_1\right)-1} \int_{-\infty}^{+\infty} A_l(m - m_1, \epsilon) s^q R_l \cdot \left(\sqrt{-1}m_1\right) \times \left(\omega_1\left(s^{1/k_1}, m_1\right) - \omega_2\left(s^{1/k_1}, m_1\right)\right) \frac{ds}{s} dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \mathrm{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} |\epsilon|^{d_{l,q}+k_1(q-p)} \cdot \|\omega_1(\tau, m) - \omega_2(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{165}$$

for $1 \le l \le D - 1$, $1 \le q \le \delta_l$, and $1 \le p \le q - 1$ in a row with

$$\left\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} C_1(m - m_1, \epsilon)(\omega_1(\tau, m_1) - \omega_2(\tau, m_1)) dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left| \frac{1}{R_D\left(\sqrt{-1}m\right)} \right| \cdot C_2 \mathbf{C}_{1,\epsilon_0} \|\omega_1(\tau, m) - \omega_2(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \tag{166}$$

Upper estimates for the rear part of $\mathscr{H}_\epsilon(\omega_1) - \mathscr{H}_\epsilon(\omega_2)$ ask some groundwork. Indeed, according to the classical identity $ab - cd = (a - c)b + c(b - d)$, we reshape

$$\begin{aligned}
\Delta(\tau, m) :={}& \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} Q_1\left(\sqrt{-1}(m - m_1)\right) \omega_1 \cdot \left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) \\
&\times Q_2\left(\sqrt{-1}m_1\right) \omega_1\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1 \\
&- \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} Q_1\left(\sqrt{-1}(m - m_1)\right) \cdot \omega_2\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) \\
&\times Q_2\left(\sqrt{-1}m_1\right) \omega_2\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1 \\
={}& \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} \left[ Q_1\left(\sqrt{-1}(m - m_1)\right) \cdot \left[\omega_1\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) - \omega_2\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right)\right] \right. \\
&\times Q_2\left(\sqrt{-1}m\right) \omega_1\left(s^{1/k_1}, m_1\right) \\
&+ Q_1\left(\sqrt{-1}(m - m_1)\right) \omega_2\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) \\
&\left. \cdot Q_2\left(\sqrt{-1}m_1\right) \times \left[\omega_1\left(s^{1/k_1}, m_1\right) - \omega_2\left(s^{1/k_1}, m_1\right)\right] \right] \cdot \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1.
\end{aligned} \tag{167}$$

Keeping in mind the factorization (157) with (158), Proposition 11 sparks a constant $C_3 > 0$ with

$$\left\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} Q_1\left(\sqrt{-1}(m - m_1)\right) \times \left[\omega_1\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right) - \omega_2\left(\left(\tau^{k_1} - s\right)^{1/k_1}, m - m_1\right)\right] \times Q_2\left(\sqrt{-1}m\right) \omega_1\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1} - s\right)s} ds dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_3 \|\omega_1(\tau, m) - \omega_2(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \cdot \|\omega_1(\tau, m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{168}$$

$$\begin{aligned}&\left\|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}}Q_1\left(\sqrt{-1}(m-m_1)\right)\right.\\&\quad\cdot\omega_2\left(\left(\tau^{k_1}-s\right)^{1/k_1},m-m_1\right)Q_2\left(\sqrt{-1}m_1\right)\\&\quad\times\left[\omega_1\left(s^{1/k_1},m_1\right)-\omega_2\left(s^{1/k_1},m_1\right)\right]\Big]\\&\quad\left.\cdot\frac{1}{\left(\tau^{k_1}-s\right)s}dsdm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\le\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_3\|\omega_1(\tau,m)-\omega_2(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\cdot\|\omega_2(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned}\tag{169}$$

The remodeling (167) of $\Delta(\tau,m)$ together with (168) and (169) leads to

$$\begin{aligned}&\|\Delta(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\le\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\\&\qquad\cdot C_3\left(\|\omega_1(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}+\|\omega_2(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\right)\\&\qquad\times\|\omega_1(\tau,m)-\omega_2(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\\&\quad\le\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_3 2\varpi_2\|\omega_1(\tau,m)-\omega_2(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.\end{aligned}\tag{170}$$

We enclose the constants $\epsilon_0>0$, $C_{1,\epsilon_0}>0$, and $c_{Q_1,Q_2}\in\mathbb{C}^*$ in the vicinity of the origin allowing the next inequality

$$\begin{aligned}&\left(\sum_{q=1}^{\delta_D-1}\left|a_{q,\delta_D}\right|\left[\frac{1}{\Gamma\left(d_{D,q}/k_1\right)}k_1^q\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\epsilon_0^{(\delta_D-q)k_1}\right.\right.\\&\quad+\sum_{1\le p\le q-1}\left|A_{q,p}\right|\frac{1}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)}\\&\quad\left.\left.\cdot k_1^p\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\epsilon_0^{(\delta_D-p)k_1}\right]\right)\\&\quad+\left[\sum_{1\le p\le\delta_D-1}\left|A_{\delta_D,p}\right|\frac{k_1^p}{\Gamma(\delta_D-p)}\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\epsilon_0^{(\delta_D-p)k_1}\right]\\&\quad+\sum_{l=1}^{D-1}\epsilon_0^{\Delta_l-d_l}\left[\sum_{q=1}^{\delta_l}\left|a_{q,\delta_l}\right|\left[\frac{k_1^q}{\Gamma\left(d_{l,q}/k_1\right)}\frac{1}{(2\pi)^{1/2}}\right.\right.\\&\quad\cdot\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2A_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}}\\&\quad+\sum_{1\le p\le q-1}\left|A_{q,p}\right|\frac{k_1^p}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)}\frac{1}{(2\pi)^{1/2}}\\&\quad\left.\left.\cdot\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2A_{l,\epsilon_0}C_1M_{k_1,\delta_D}\epsilon_0^{d_{l,q}+k_1(q-p)}\right]\right]\\&\quad+\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2C_{1,\epsilon_0}\\&\quad+\left|c_{Q_1,Q_2}\right|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}2\varpi_2\\&\le1/2.\end{aligned}\tag{171}$$

The merging of the above bounds (161)–(166) and (170) subjected to (171) triggers the $1/2-$Lipschitz attribute of $\mathscr{H}_\epsilon$. Notice that the foremost five blocks of the left handside of (171) can be taken small scaled since they contain positive powers of $\varepsilon_0$ due to the constraint (41) imposed on (37) and its two tail terms can be downsized provided that the positive constants $C_{1,\epsilon_0}$ and $c_{Q_1,Q_2}$ are chosen close to the origin.

In the closing part of the proof, we fix the radius $\varpi_2>0$ and select the quantities $\epsilon_0>0$, $C_{1,\epsilon_0}>0$ together with $c_{Q_1,Q_2}\in\mathbb{C}^*$ close enough to 0 that conforms both (160) and (171). For these values, the map $\mathscr{H}_\epsilon$ is endowed with both inclusion and shrinking properties (137) and (138) for all $\epsilon\in D_{\epsilon_0}\setminus\{0\}$. Proposition 13 follows. □

The forthcoming proposition displays a solution to the first convolution Equation (83) shaped in the Banach spaces described in Definition 4.

**Proposition 14.** *Let us choose an appropriate inner radius $r_{Q,R_D}>0$ and aperture $\eta_{Q,R_D}>0$ of the sector $S_{Q,R_D}$ together with an unbounded sector $S_{d_l}$ and radius $\rho>0$ that conforms the requirements of Lemma 12. Then, a radius $\epsilon_0>0$ and constants $C_{1,\epsilon_0}>0$, $c_{Q_1,Q_2}\in\mathbb{C}^*$ can be pinpointed sufficiently close to 0 together with a proper radius $\varpi_2>0$ in a manner that for all $\epsilon\in D_{\epsilon_0}\setminus\{0\}$, a unique solution $\omega_{2,d_l}(\tau,m,\epsilon)$ to (83) exists such that*

*(i) The map $(\tau,m)\mapsto\omega_{2,d_l}(\tau,m,\epsilon)$ appertains to $F^{d_l}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ under the constraint*

$$\sup_{\epsilon\in D_{\epsilon_0}\setminus\{0\}}\left\|\omega_{2,d_l}(\tau,m,\epsilon)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\le\varpi_2.\tag{172}$$

*The partial map $\epsilon\mapsto\omega_{2,d_l}(\tau,m,\epsilon)$ stands for an analytic map from $D_{\epsilon_0}\setminus\{0\}$ into $\mathbb{C}$, for any prescribed $\tau\in S_{d_l}\cup D_\rho$ and $m\in\mathbb{R}$.*

*Proof.* We take the constants $\epsilon_0>0$, $C_{1,\epsilon_0}>0$, $c_{Q_1,Q_2}\in\mathbb{C}^*$ together with $\varpi_2>0$ reached in Proposition 13. We observe that the closed ball $\bar{B}_{\varpi_2}\subset F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ represents a complete metric space for the distance $d(x,y)=\|x-y\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$. Proposition 4 claims that $\mathscr{H}_\epsilon$ induces a contractive map from $(\bar{B}_{\varpi_2}, d)$ into itself. It follows from the classical Banach fixed-point theorem that $\mathscr{H}_\epsilon$ possesses a unique fixed point $\omega_{2,d_1}(\tau, m, \epsilon)$ inside the ball $\bar{B}_{\varpi_2}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, meaning that

$$\mathscr{H}_\epsilon\left(\omega_{2,d_1}(\tau, m, \epsilon)\right) = \omega_{2,d_1}(\tau, m, \epsilon), \tag{173}$$

holds. Furthermore, the map $\omega_{2,d_1}(\tau, m, \varepsilon)$ relies analytically on $\epsilon$ since $\mathscr{H}_\epsilon$ does on the domain $D_{\epsilon_0} \setminus \{0\}$. On the other hand, we check that the convolution Equation (83) can be rearranged as the Equation (173) by shifting the term

$$\left(k_1 \tau^{k_1}\right)^{\delta_D} R_D\left(\sqrt{-1}m\right) \omega_{2,d_1}(\tau, m, \epsilon), \tag{174}$$

from the right to the left handside of (83) and dividing by the resulting equation by the map $P_m(\tau)$ given by (127). As a result, the unique fixed point $\omega_{2,d_1}(\tau, m, \epsilon)$ of $\mathscr{H}_\epsilon$ enclosed in $\bar{B}_{\varpi_2}$ precisely solves (83). The result follows. □

## 6. Building Up a Solution to the Second Convolution Equation (84) with (85)

In this section, we cook up a unique solution to the auxiliary convolution equation reached in (84) with (85) inside the Banach spaces described in Definition 4.

The roadmap follows the one of the previous section and consists in recasting (84) with (85) into a fixed-point equation for a certain nonlinear map $\mathscr{G}_\epsilon$, stated in Proposition 16.

The map $\mathscr{G}_\epsilon$ is set up as follows. We mind the map $\omega_{2,d_1}(\tau, m, \epsilon)$ stemming from Proposition 14 and the polynomial $P_m(\tau)$ displayed in (127). Let

$$\begin{aligned}
&\mathscr{G}_\epsilon(\omega(\tau, m)) \\
&:= \left(\sum_{q=1}^{\delta_D-1} a_{q,\delta_D}\left[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{D,q}/k_1)} \cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1} k_1^q s^q \omega\left(s^{1/k_1}, m\right)\frac{ds}{s}\right.\right. \\
&\quad + \sum_{1\le p\le q-1} A_{q,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{D,q}+k_1(q-p)/k_1)} \\
&\quad \left.\cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1} k_1^p s^p \omega\left(s^{1/k_1}, m\right)\frac{ds}{s}\right] \\
&\quad \left.\times R_D\left(\sqrt{-1}m\right)\right) + \left[\sum_{1\le p\le \delta_D-1} A_{\delta_D,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(\delta_D-p)}\right. \\
&\quad \left.\cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\delta_D-p-1} k_1^p s^p \omega\left(s^{1/k_1}, m\right)\frac{ds}{s} \times R_D\left(\sqrt{-1}m\right)\right] \\
&\quad + \left(\delta_D \sum_{q=1}^{\delta_D-1} a_{q,\delta_D-1}\left[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{D,q}/k_1)}\right.\right. \\
&\quad \cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1} k_1^q s^q \omega_{2,d_1}\left(s^{1/k_1}, m, \epsilon\right)\frac{ds}{s} \\
&\quad + \sum_{1\le p\le q-1} A_{q,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{D,q}+k_1(q-p)/k_1)} \\
&\quad \left.\cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1} k_1^p s^p \omega_{2,d_1}\left(s^{1/k_1}, m, \epsilon\right)\frac{ds}{s}\right] \\
&\quad \left.\times R_D\left(\sqrt{-1}m\right)\right) + \sum_{l=1}^{D-1} \epsilon^{\Delta_l - d_l} \\
&\quad \cdot \left[\left(\sum_{q=1}^{\delta_l} a_{q,\delta_l}\left[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{l,q}/k_1)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}\right.\right.\right. \\
&\quad \cdot \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_l(m-m_1, \epsilon) k_1^q s^q \times R_l\left(\sqrt{-1}m_1\right) \\
&\quad \cdot \omega\left(s^{1/k_1}, m_1\right)\frac{ds}{s}dm_1 + \sum_{1\le p\le q-1} A_{q,p} \\
&\quad \cdot \frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{l,q}+k_1(q-p)/k_1)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}+k_1(q-p)/k_1)-1} \\
&\quad \times \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{+\infty} A_l(m-m_1, \epsilon) k_1^p s^p R_l\left(\sqrt{-1}m_1\right) \\
&\quad \left.\left.\cdot \omega\left(s^{1/k_1}, m_1\right)\frac{ds}{s}dm_1\right]\right) + \left(\delta_l \sum_{q=1}^{\delta_l-1} a_{q,\delta_l-1}\right. \\
&\quad \cdot \left[\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{l,q}/k_1)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}\frac{1}{(2\pi)^{1/2}}\right. \\
&\quad \cdot \int_{-\infty}^{+\infty} A_l(m-m_1, \epsilon) k_1^q s^q R_l\left(\sqrt{-1}m_1\right) \times \omega_{2,d_1}\left(s^{1/k_1}, m_1, \epsilon\right) \\
&\quad \cdot \frac{ds}{s}dm_1 + \sum_{1\le p\le q-1} A_{q,p}\frac{\tau^{k_1}}{P_m(\tau)\Gamma(d_{l,q}+k_1(q-p)/k_1)} \\
&\quad \cdot \int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}+k_1(q-p)/k_1)-1} \times \frac{1}{(2\pi)^{1/2}} \\
&\quad \cdot \int_{-\infty}^{+\infty} A_l(m-m_1, \epsilon) k_1^p s^p R_l\left(\sqrt{-1}m_1\right)\omega_{2,d_1} \\
&\quad \left.\left.\left.\cdot \left(s^{1/k_1}, m_1, \epsilon\right)\frac{ds}{s}dm_1\right]\right)\right] + \mathscr{A}_{\mathscr{G}_\epsilon}(\tau, m, \epsilon),
\end{aligned} \tag{175}$$

where

$$\begin{aligned}
\mathscr{A}_{\mathscr{G}_\epsilon}(\tau, m, \epsilon) &:= \frac{\mathscr{F}_1(\tau, m, \epsilon)}{P_m(\tau)} + \frac{1}{P_m(\tau)(2\pi)^{1/2}} \\
&\quad \cdot \int_{-\infty}^{+\infty} B_1(m-m_1, \epsilon)\omega(\tau, m_1)dm_1 \\
&\quad + \frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty} B_2(m-m_1, \epsilon)\omega_{2,d_1} \\
&\quad \cdot (\tau, m_1, \epsilon)dm_1 + c_{P_1P_2}\frac{1}{P_m(\tau)(2\pi)^{1/2}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} P_1\left(\sqrt{-1}(m-m_1)\right)\omega\left(\left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1\right) \\
&\times P_2\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1}, m_1, \epsilon\right)\frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 \\
&+ c_{P_3P_4}\frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1}\int_0^{\tau^{k_1}} P_3\left(\sqrt{-1}(m-m_1)\right) \\
&\cdot \omega\left(\left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1\right) \times P_4\left(\sqrt{-1}m_1\right)\omega\left(s^{1/k_1}, m_1\right) \\
&\cdot \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 + c_{P_5P_6}\frac{1}{P_m(\tau)(2\pi)^{1/2}}\int_{-\infty}^{+\infty}\tau^{k_1} \\
&\cdot \int_0^{\tau^{k_1}} P_5\left(\sqrt{-1}(m-m_1)\right)\omega_{2,d_1}\left(\left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1, \epsilon\right) \\
&\times P_6\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1}, m_1, \epsilon\right)\frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1.
\end{aligned} \tag{176}
$$

In the next proposition, we discuss the $1/2-$Lipschitz feature of $\mathcal{G}_\epsilon$ on some well-chosen ball in the Banach spaces depicted in Definition 4.

**Proposition 15.** *Let a timely inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector $S_{Q,R_D}$ in a row with an unbounded sector $S_{d_1}$ and radius $\rho > 0$ chosen to fulfill the specifications of Lemma 12. We also take for granted the additional condition (136) required for the sector $S_{d_1}$ and the disc $D_\rho$.*

*Then, one can target a small radius $\epsilon_0 > 0$ along with constants $B_{j,\epsilon_0} > 0$, $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $j = 1, 2$ and $k = 1, 3, 5$ proximate to 0, coupled to a fitted radius $\varpi_1 > 0$ in a way that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the map $\mathcal{G}_\epsilon$ boasts the next two properties*

*(i) $\mathcal{G}_\epsilon$ maps $\bar{B}_{\varpi_1}$ into itself, where $\bar{B}_{\varpi_1}$ stands for the closed ball of radius $\varpi_1$ centered at 0 in the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*

*(ii) The norm downsizing condition*

$$
\|\mathcal{G}_\epsilon(\omega_1) - \mathcal{G}_\epsilon(\omega_2)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{2}\|\omega_1 - \omega_2\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{177}
$$

*holds whenever $\omega_1, \omega_2 \in F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.*

*Proof.* We heed the first item asserting the inclusion. We fix some real number $\varpi_1 > 0$ and pick up an element $\omega(\tau, m)$ in $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$, for $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, with

$$
\|\omega\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \varpi_1. \tag{178}
$$

Concrete bounds are presented for each piece of the map $\mathcal{G}_\epsilon$ applied to $\omega$.

The estimates for the first three blocks of $\mathcal{G}_\epsilon$ are merely the same as the ones obtained in (140)–(142). Namely, owing to Proposition 8 and Lemma 12, we observe that

$$
\begin{aligned}
&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1} s^q R_D\left(\sqrt{-1}m\right)\omega\left(s^{1/k_1}, m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-q)k_1}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},
\end{aligned} \tag{179}
$$

for $1 \le q \le \delta_D - 1$ along with

$$
\begin{aligned}
&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1} s^p R_D\left(\sqrt{-1}m\right)\omega\left(s^{1/k_1}, m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},
\end{aligned} \tag{180}
$$

for $1 \le p \le q-1$ with $1 \le q \le \delta_D - 1$ and

$$
\begin{aligned}
&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\delta_D-p-1} s^p R_D\left(\sqrt{-1}m\right)\omega\left(s^{1/k_1}, m\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)},
\end{aligned} \tag{181}
$$

as long as $1 \le p \le \delta_D - 1$.

The next two pieces of $\mathcal{G}_\epsilon$ follow from Proposition 8 and Lemma 12 together with the estimates (172) reached in Proposition 14. Indeed, we arrive at

$$
\begin{aligned}
&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1} s^q R_D\left(\sqrt{-1}m\right)\omega_{2,d_1}\left(s^{1/k_1}, m, \epsilon\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-q)k_1}\|\omega_{2,d_1}(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-q)k_1}\varpi_2
\end{aligned} \tag{182}
$$

for $1 \le q \le \delta_D - 1$ in a row with

$$
\begin{aligned}
&\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1} s^p \right. \\
&\qquad \left. \cdot R_D\left(\sqrt{-1}m\right)\omega_{2,d_1}\left(s^{1/k_1}, m, \epsilon\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\|\omega_{2,d_1}(\tau,m,\epsilon)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \le \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\varpi_2
\end{aligned} \tag{183}
$$

for $1 \le p \le q-1$ with $1 \le q \le \delta_D - 1$.

The estimates for the following two components of $\mathcal{G}_\epsilon$ simply recast the ones obtained in (148) and (152). Indeed,

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^q \times R_l\left(\sqrt{-1}m_1\right)\omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \quad (184)$$

for $1 \le q \le \delta_l$ and $1 \le l \le D-1$ in parallel with

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{d_{l,q}+k_1(q-p)/k_1-1}\times\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^pR_l \cdot\left(\sqrt{-1}m_1\right)\omega\left(s^{1/k_1},m_1\right)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)} \cdot\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \quad (185)$$

for $1 \le p \le q-1$ and $1 \le q \le \delta_l$ with $1 \le l \le D-1$. Furthermore, the two ensuing constituents of $\mathcal{G}_\epsilon$ mirror the one reached in (148) and (152) and draw on the estimates (172) from Proposition 14. Namely,

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^q \times R_l\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D} \cdot|\epsilon|^{d_{l,q}}\left\|\omega_{2,d_1}(\tau,m,\varepsilon)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}\varpi_2, \quad (186)$$

for $1 \le q \le \delta_l - 1$ and $1 \le l \le D-1$ in tandem with

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}+k_1(q-p)/k_1)-1}\times\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^pR_l \cdot\left(\sqrt{-1}m_1\right)\omega_{2,d_1}\left(s^{1/k_1},m_1,\epsilon\right)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)} \cdot\left\|\omega_{2,d_1}(\tau,m,\epsilon)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}+k_1(q-p)}\varpi_2, \quad (187)$$

provided that $1 \le p \le q-1$ and $1 \le q \le \delta_l - 1$ with $1 \le l \le D-1$.

The next element of $\mathcal{G}_\epsilon$ we pay regard is $\mathcal{F}_1(\tau,m,\epsilon)/P_m(\tau)$ and is displayed in (176). Its bounds are obtained in a similar way as the ones reached in (155). Indeed,

$$\left\|\frac{\mathcal{F}_1(\tau,m,\epsilon)}{P_m(\tau)}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|\epsilon_0 \times\sup_{x\ge0}e^{-\nu x^{k_1}}\left(1+x^{2k_1}\right)\sum_{j_1\in J_1}\mathrm{F}_{1,j_1,\epsilon_0}\epsilon_0^{j_1-1}x^{j_1-1}, \quad (188)$$

which can be subsided close to 0 provided that $\epsilon_0 > 0$ is tiny enough since $0 \notin J_1$.

We handle the second and third pieces of $\mathcal{A}_{\mathcal{G}_\epsilon}(\tau,m,\epsilon)$. Paying heed to (153) and the bounds (55), Proposition 10 kindles

$$\left\|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}B_1(m-m_1,\epsilon)\omega(\tau,m_1)dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right| \cdot C_2\|B_1(m,\epsilon)\|_{(\beta,\mu)}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right| \cdot C_2\mathrm{B}_{1,\epsilon_0}\|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \quad (189)$$

and bearing in mind the estimates (172) from Proposition 14,

$$\left\|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}B_2(m-m_1,\epsilon)\omega_{2,d_1}(\tau,m_1,\epsilon)dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2\|B_2(m,\varepsilon)\|_{(\beta,\mu)} \cdot\left\|\omega_{2,d_1}(\tau,m,\epsilon)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2\mathrm{B}_{2,\epsilon_0}\varpi_2, \quad (190)$$

ensues.

Thanks to factorization (157) with (158) and the bounds (172) from Proposition 14, we can apply Proposition 11 in order to address the last three terms of $\mathscr{A}_{\mathscr{G}_\epsilon}(\tau, m, \epsilon)$. Namely,

$$\begin{aligned}&\Bigg\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} P_1\left(\sqrt{-1}(m-m_1)\right)\omega \\ &\cdot \left(\left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1\right) \times P_2\left(\sqrt{-1}m_1\right)\omega_{2,d_1} \\ &\cdot \left(s^{1/k_1}, m_1, \epsilon\right) \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 \Bigg\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\cdot \left\|\omega_{2,d_1}(\tau,m,\epsilon)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \|\omega(\tau,m)\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \varpi_2,\end{aligned} \tag{191}$$

together with

$$\begin{aligned}&\Bigg\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} P_3\left(\sqrt{-1}(m-m_1)\right) \\ &\cdot \omega\left(\left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1\right) \\ &\times P_4\left(\sqrt{-1}m_1\right)\omega\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 \Bigg\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \|\omega(\tau,m)\|^2_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}\end{aligned} \tag{192}$$

as well as

$$\begin{aligned}&\Bigg\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} P_5\left(\sqrt{-1}(m-m_1)\right)\omega_{2,d_1} \\ &\cdot \left(\left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1, \epsilon\right) \times P_6\left(\sqrt{-1}m_1\right)\omega_{2,d_1} \\ &\cdot \left(s^{1/k_1}, m_1, \epsilon\right) \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 \Bigg\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \left\|\omega_{2,d_1}(\tau,m,\epsilon)\right\|^2_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\ &\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \varpi_2^2.\end{aligned} \tag{193}$$

We pin down the constants $\epsilon_0 > 0$ and $\mathrm{B}_{j,\epsilon_0} > 0$, $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $j = 1, 2$ and $k = 1, 3, 5$ proximate to 0 together with a suitable radius $\varpi_1 > 0$ in a way that the next inequality

$$\begin{aligned}&\left(\sum_{q=1}^{\delta_D-1} \left|a_{q,\delta_D}\right| \left[\frac{1}{\Gamma\left(d_{D,q}/k_1\right)} k_1^q \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-q)k_1}\varpi_1 \right.\right. \\ &+ \sum_{1\leq p\leq q-1} \left|A_{q,p}\right| \frac{1}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)} k_1^p \\ &\left.\left.\cdot \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-p)k_1}\varpi_1\right]\right) \\ &+ \sum_{1\leq p\leq \delta_D-1} \left|A_{\delta_D,p}\right| \frac{k_1^p}{\Gamma(\delta_D-p)} \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-p)k_1}\varpi_1 \\ &+ \left(\delta_D \sum_{q=1}^{\delta_D-1} \left|a_{q,\delta_D-1}\right| \left[\frac{1}{\Gamma\left(d_{D,q}/k_1\right)} k_1^q \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \right.\right. \\ &\cdot \epsilon_0^{(\delta_D-q)k_1}\varpi_2 + \sum_{1\leq p\leq q-1} \left|A_{q,p}\right| \frac{1}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)} \\ &\left.\left.\cdot k_1^p \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-p)k_1}\varpi_2\right]\right) + \sum_{l=1}^{D-1} \epsilon_0^{\Delta_l-d_l} \\ &\cdot \left[\sum_{q=1}^{\delta_l} \left|a_{q,\delta_l}\right| \left[\frac{k_1^q}{\Gamma\left(d_{D,q}/k_1\right)} \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\right.\right. \\ &\cdot C_2 \mathrm{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}}\varpi_1 + \sum_{1\leq p\leq q-1} \left|A_{q,p}\right| \\ &\cdot \frac{k_1^p}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)} \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \\ &\left.\cdot C_2 \mathrm{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}+k_1(q-p)}\varpi_1\right] \\ &+ \delta_l \sum_{q=1}^{\delta_l-1} a_{q,\delta_l-1} \left[\frac{k_1^q}{\Gamma\left(d_{l,q}/k_1\right)} \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\right. \\ &\cdot C_2 \mathbf{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}}\varpi_2 + \sum_{1\leq p\leq q-1} \left|A_{q,p}\right| \\ &\cdot \frac{k_1^p}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)} \frac{1}{(2\pi)^{1/2}} \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \\ &\left.\left.\cdot C_2 \mathbf{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}+k_1(q-p)}\varpi_2\right]\right] + \mathbb{A}_{\mathscr{G}} \leq \varpi_1,\end{aligned} \tag{194}$$

holds where

$$\begin{aligned}\mathbb{A}_{\mathscr{G}} = &\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right| \epsilon_0 \\ &\times \sup_{x\geq 0} e^{-\nu x^{k_1}} \left(1+x^{2k_1}\right) \sum_{j_1\in J_1} \mathrm{F}_{1,j_1,\epsilon_0} \epsilon_0^{j_1-1} x^{j_1-1}\end{aligned}$$

$$+\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2\mathrm{B}_{1,\epsilon_0}\varpi_1$$
$$+\frac{1}{(2\pi)^{1/2}}\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|C_2\mathrm{B}_{2,\epsilon_0}\varpi_2$$
$$+\left|c_{P_1,P_2}\right|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\varpi_1\varpi_2$$
$$+\left|c_{P_3,P_4}\right|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\varpi_1^2$$
$$+\left|c_{P_5,P_6}\right|\frac{1}{(2\pi)^{1/2}}\frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\varpi_2^2. \tag{195}$$

We check that all the terms on the left handside of (194) except $\mathbb{A}_{\mathcal{G}}$ can be tapered off since they contain positive powers of $\epsilon_0>0$ in particular due to the constraint (41). Besides, the constant $\mathbb{A}_{\mathcal{G}}$ can be lessen provided that the constants $\epsilon_0$ and $\mathrm{B}_{j,\epsilon_0}$, $c_{P_k,P_{k+1}}$, for $j=1,2$ and $k=1,3,5$ are taken in the vicinity of 0.

At last, stacking up all the above bounds ((179)–(193), under the contingency (194) yield that $\mathcal{G}_\epsilon$ maps $\bar{B}_{\varpi_1}$ into itself.

In the second part of the proof, we address the second item of Proposition 15. Let $\omega_1,\omega_2$ be elements of the ball $\bar{B}_{\varpi_1}$ of the space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ with radius $\varpi_1>0$ chosen as in the first part of the proof.

We provide norm estimates for each part of the difference $\mathcal{G}_\epsilon(\omega_1)-\mathcal{G}_\epsilon(\omega_2)$. The bounds for the foremost five blocks of the difference are barely the ones found in (161)–(165). Namely,

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}/k_1)-1}s^qR_D\left(\sqrt{-1}m\right)\cdot\left(\omega_1\left(s^{1/k_1},m\right)-\omega_2\left(s^{1/k_1},m\right)\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\le\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-q)k_1}\left\|\omega_1(\tau,m)-\omega_2(\tau,m)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{196}$$

for $1\le q\le\delta_D-1$ along with

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{D,q}+k_1(q-p)/k_1)-1}s^pR_D\left(\sqrt{-1}m\right)\times\left(\omega_1\left(s^{1/k_1},m\right)-\omega_2\left(s^{1/k_1},m\right)\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\le\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\left\|\omega_1(\tau,m)-\omega_2(\tau,m)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{197}$$

for $1\le p\le q-1$ with $1\le q\le\delta_D-1$ and

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{\delta_D-p-1}s^pR_D\left(\sqrt{-1}m\right)\cdot\left(\omega_1\left(s^{1/k_1},m\right)-\omega_2\left(s^{1/k_1},m\right)\right)\frac{ds}{s}\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\le\frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}|\epsilon|^{(\delta_D-p)k_1}\left\|\omega_1(\tau,m)-\omega_2(\tau,m)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{198}$$

as long as $1\le p\le\delta_D-1$. Furthermore,

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}/k_1)-1}\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^qR_l\left(\sqrt{-1}m_1\right)\times\left(\omega_1\left(s^{1/k_1},m_1\right)-\omega_2\left(s^{1/k_1},m_1\right)\right)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\le\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathbf{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}|\epsilon|^{d_{l,q}}\cdot\left\|\omega_1(\tau,m)-\omega_2(\tau,m)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{199}$$

holds for $1\le l\le D-1$ and $1\le q\le\delta_l$ together with

$$\left\|\frac{\tau^{k_1}}{P_m(\tau)}\int_0^{\tau^{k_1}}\left(\tau^{k_1}-s\right)^{(d_{l,q}+k_1(q-p)/k_1)-1}\cdot\int_{-\infty}^{+\infty}A_l(m-m_1,\epsilon)s^pR_l\left(\sqrt{-1}m_1\right)\times\left(\omega_1\left(s^{1/k_1},m_1\right)-\omega_2\left(s^{1/k_1},m_1\right)\right)\frac{ds}{s}dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\le\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}C_2\mathrm{A}_{l,\epsilon_0}C_1M_{k_1,\delta_D}\cdot|\epsilon|^{d_{l,q}+k_1(q-p)}\left\|\omega_1(\tau,m)-\omega_2(\tau,m)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}, \tag{200}$$

for $1\le l\le D-1$, $1\le q\le\delta_l$ and $1\le p\le q-1$. Besides, bounds for the sixth piece of $\mathcal{G}_\epsilon(\omega_1)-\mathcal{G}_\epsilon(\omega_2)$ result from (189) and are written

$$\left\|\frac{1}{P_m(\tau)}\int_{-\infty}^{+\infty}B_1(m-m_1,\epsilon)(\omega_1(\tau,m_1)-\omega_2(\tau,m_1))dm_1\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$$
$$\le\frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}}\max_{m\in\mathbb{R}}\left|\frac{1}{R_D\left(\sqrt{-1}m\right)}\right|\cdot C_2\mathbf{B}_{1,\epsilon_0}\left\|\omega_1(\tau,m)-\omega_2(\tau,m)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}. \tag{201}$$

The treatment of the seventh piece of $\mathcal{G}_{\epsilon}(\omega_1) - \mathcal{G}_{\epsilon}(\omega_2)$ springs from (191). Indeed,

$$\begin{aligned}
&\left\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} P_1\left(\sqrt{-1}(m-m_1)\right) \right. \\
&\quad \times \left( \omega_1\left( \left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1 \right) \right. \\
&\quad \left. - \omega_2\left( \left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1 \right) \right) \times P_2\left(\sqrt{-1}m_1\right) \omega_{2,d_1} \\
&\quad \left. \cdot \left(s^{1/k_1}, m_1, \epsilon\right) \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \left\| \omega_1(\tau,m) - \omega_2(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\quad \cdot \left\| \omega_{2,d_1}(\tau,m,\epsilon) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\leq \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \left\| \omega_1(\tau,m) - \omega_2(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \varpi_2.
\end{aligned} \tag{202}$$

The hindmost term of the difference $\mathcal{G}_{\epsilon}(\omega_1) - \mathcal{G}_{\epsilon}(\omega_2)$ can be processed in a similar way as for the difference (167) given by (170). Namely,

$$\begin{aligned}
&\left\| \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \int_0^{\tau^{k_1}} P_3\left(\sqrt{-1}(m-m_1)\right) \right. \\
&\quad \cdot \omega_1\left( \left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1 \right) \times P_4\left(\sqrt{-1}m_1\right) \\
&\quad \cdot \omega_1\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 - \frac{1}{P_m(\tau)} \int_{-\infty}^{+\infty} \tau^{k_1} \\
&\quad \cdot \int_0^{\tau^{k_1}} P_3\left(\sqrt{-1}(m-m_1)\right) \omega_2\left( \left(\tau^{k_1}-s\right)^{1/k_1}, m-m_1 \right) \\
&\quad \left. \times P_4\left(\sqrt{-1}m_1\right) \omega_2\left(s^{1/k_1}, m_1\right) \frac{1}{\left(\tau^{k_1}-s\right)s} ds dm_1 \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_3 \\
&\quad \cdot \left( \left\| \omega_1(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} + \left\| \omega_2(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \right) \\
&\quad \times \left\| \omega_1(\tau,m) - \omega_2(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \\
&\leq \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_3 2\varpi_1 \left\| \omega_1(\tau,m) - \omega_2(\tau,m) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}.
\end{aligned} \tag{203}$$

We skirt the constants $\epsilon_0 > 0$, $\mathbf{B}_{1,\epsilon_0} > 0$, and $c_{P_1P_2} \in \mathbb{C}^*$, $c_{P_3P_4} \in \mathbb{C}^*$ nearby the origin in a manner that the next inequality

$$\begin{aligned}
&\left( \sum_{q=1}^{\delta_D-1} \left|a_{q,\delta_D}\right| \left[ \frac{1}{\Gamma\left(d_{D,q}/k_1\right)} k_1^q \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-q)k_1} \right. \right. \\
&\quad + \sum_{1\leq p\leq q-1} \left|A_{q,p}\right| \frac{1}{\Gamma\left(d_{D,q}+k_1(q-p)/k_1\right)} \\
&\quad \left. \left. \cdot k_1^p \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-p)k_1} \right] \right) + \sum_{1\leq p\leq \delta_D-1} \left|A_{\delta_D,p}\right| \\
&\quad \cdot \frac{k_1^p}{\Gamma\left(\delta_D-p\right)} \frac{C_1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \epsilon_0^{(\delta_D-p)k_1} \\
&\quad + \sum_{l=1}^{D-1} \epsilon_0^{\Delta_l-d_l} \left[ \sum_{q=1}^{\delta_l} \left|a_{q,\delta_l}\right| \left[ \frac{k_1^q}{\Gamma\left(d_{l,q}/k_1\right)} \frac{1}{(2\pi)^{1/2}} \right. \right. \\
&\quad \cdot \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \mathbf{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}} + \sum_{1\leq p\leq q-1} \left|A_{q,p}\right| \\
&\quad \cdot \frac{k_1^p}{\Gamma\left(d_{l,q}+k_1(q-p)/k_1\right)} \frac{1}{(2\pi)^{1/2}} \\
&\quad \left. \left. \cdot \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} C_2 \mathbf{A}_{l,\epsilon_0} C_1 M_{k_1,\delta_D} \epsilon_0^{d_{l,q}+k_1(q-p)} \right] \right] + \mathbb{S}_{\mathcal{G}} \\
&\leq \frac{1}{2},
\end{aligned} \tag{204}$$

holds where

$$\begin{aligned}
\mathbb{S}_{\mathcal{G}} = &\frac{1}{(2\pi)^{1/2}} \frac{1}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \max_{m\in\mathbb{R}} \left| \frac{1}{R_D\left(\sqrt{-1}m\right)} \right| C_2 \mathbf{B}_{1,\epsilon_0} \\
&+ \left|c_{P_1,P_2}\right| \frac{1}{(2\pi)^{1/2}} \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} \varpi_2 \\
&+ \left|c_{P_3,P_4}\right| \frac{1}{(2\pi)^{1/2}} \frac{C_3}{C_P\left(r_{Q,R_D}\right)^{1/k_1\delta_D}} 2\varpi_1.
\end{aligned} \tag{205}$$

We notice that all the terms appearing in the left hand-side of (204) excluding $\mathbb{S}_{\mathcal{G}}$ can be dwindled since they involve positive powers of $\epsilon_0$ according to the constraints (41). Furthermore, the term $\mathbb{S}_{\mathcal{G}}$ can be depleted whenever the constants $\mathbf{B}_{1,\epsilon_0} > 0$ and $c_{P_1P_2} \in \mathbb{C}^*$, $c_{P_3P_4} \in \mathbb{C}^*$ are taken close to 0.

In the end, the coupling of all the above bounds (196)–(203)under condition (204) triggers the shrinking feature (177) for the map $\mathcal{G}_{\epsilon}$.

In conclusion, we select the radius $\varpi_1 > 0$ and pinpoint the constants $\epsilon_0 > 0$, $\mathbf{B}_{j,\epsilon_0} > 0$, for $j = 1, 2$, along with $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $k = 1, 3, 5$ nearby the origin, in a way they obey both (194) and (204). These values taken for granted, the map $\mathcal{G}_{\epsilon}$ fulfills both inclusion and shrinking properties described in the items of Proposition 15. □

The oncoming proposition provides a solution to the second convolution equation (84) with (85) crafted in the Banach spaces displayed in Definition 4.

**Proposition 16.** *Consider an appropriate inner radius $r_{Q,R_D} > 0$ and aperture $\eta_{Q,R_D} > 0$ of the sector $S_{Q,R_D}$ together with an unbounded sector $S_{d_1}$ and radius $\rho > 0$ that respect the requirements of Lemma 12. Then, a radius $\epsilon_0 > 0$ along with* *constants* $B_{j,\epsilon_0} > 0$, *for* $j = 1, 2$ *and* $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, *for* $k = 1, 3, 5$ *can be pinned down nearby 0 together with an appropriate radius* $\varpi_1 > 0$ *in a way that for all* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, *a unique solution* $\omega_{1,d_1}(\tau, m, \epsilon)$ *to (84), (85) exists that is favoured with the next features*

(i) *the map* $(\tau, m) \mapsto \omega_{1,d_1}(\tau, m, \epsilon)$ *belongs to* $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ *under the restriction*

$$\sup_{\epsilon\in D_{\epsilon_0}\setminus\{0\}} \left\|\omega_{1,d_1}(\tau, m, \epsilon)\right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \varpi_1. \tag{206}$$

(ii) *the partial map* $\epsilon \mapsto \omega_{1,d_1}(\tau, m, \epsilon)$ *stands for an analytic map from* $D_{\epsilon_0} \setminus \{0\}$ *into* $\mathbb{C}$, *for any prescribed* $\tau \in S_{d_1} \cup D_\rho$ *and* $m \in \mathbb{R}$.

*Proof.* Let the constants $\epsilon_0 > 0$, $\mathbf{B}_{j,\epsilon_0} > 0$, for $j = 1, 2$ and $c_{P_k,P_{k+1}} \in \mathbb{C}^*$, for $k = 1, 3, 5$ together with $\varpi_1 > 0$ be fixed as in Proposition 15. The proposition 6 asserts that $\mathcal{G}_\epsilon$ induces a contractive map from the closed ball and complete space $\bar{B}_{\varpi_1}$ into itself for the distance $d(x, y) = \|x - y\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ inherited from the norm on the Banach space $F^{d_1}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$.

The classical Banach fixed point theorem then claims that $\mathcal{G}_\epsilon$ boasts a unique fixed point $\omega_{1,d_1}(\tau, m, \epsilon)$ inside the ball $\bar{B}_{\varpi_1}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. In other words,

$$\mathcal{G}_\epsilon\left(\omega_{1,d_1}(\tau, m, \epsilon)\right) = \omega_{1,d_1}(\tau, m, \epsilon), \tag{207}$$

holds. Furthermore, the map $\omega_{1,d_1}(\tau, m, \epsilon)$ depends analytically on $\epsilon$ since $\mathcal{G}_\epsilon$ itself does on the domain $D_{\epsilon_0} \setminus \{0\}$. On the other hand, we observe that the convolution equation (84) can be reorganized as the equation (207) by moving the term

$$\left(k_1\tau^{k_1}\right)^{\delta_D} R_D\left(\sqrt{-1}m\right)\omega_{1,d_1}(\tau, m, \epsilon) \tag{208}$$

from the right to the left handside of (84) and dividing by the resulting equation by the map $P_m(\tau)$ given by (127). As a result, the unique fixed point $\omega_{1,d_1}(\tau, m, \epsilon)$ of $\mathcal{G}_\epsilon$ penned in $\bar{B}_{\varpi_1}$ precisely solves (84) and (85). The result ensues. □

## 7. Building up a Finite Set of Holomorphic Solutions to the Coupling of Partial Differential Equations (66) and (67)

*7.1. Fourier-Laplace Transforms Solutions to the Pairing (66), (67).* In this section, we exhibit genuine analytic solutions expressed by means of Fourier-Laplace transforms to the coupling (66) and (67) reached at the end of Subsection 3.1.

**Proposition 17.** *For all unbounded sectors* $S_{d_1}$ *with bisecting direction* $d_1 \in \mathbb{R}$ *and disc* $D_\rho$ *that obey the demands of Lemma 12, we introduce the two partial maps*

$$\begin{aligned}(u_1, z) \mapsto U_{j,d_1}(u_1, z, \epsilon) &= \frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_{1,u_1}}}\int_{-\infty}^{+\infty}\omega_{j,d_1}(\tau, m, \epsilon) \\ &\quad\cdot \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm,\end{aligned} \tag{209}$$

*for* $j = 1, 2$, *for all* $\epsilon \in D_{\epsilon_0} \setminus \{0\}$ *where the Borel map* $\omega_{2,d_1}$ *is manufactured in Proposition 14 and solves (83), the Borel map* $\omega_{1,d_1}$ *is crafted in Proposition 16 and fulfills (84) and (85), and the radius* $\epsilon_0 > 0$ *is taken in agreement with Propositions 14 and 16 and* $L_{d_{1,u_1}} = [0,+\infty)e^{\sqrt{-1}d_{1,u_1}}$ *stands for a half-line in a direction* $d_{1,u_1} \in \mathbb{R}$ *suitably chosen and described below.*

*The maps* $U_{j,d_1}(u_1, z, \epsilon)$, $j = 1, 2$, *are endowed with the next two properties.*

*They define holomorphic functions that are bounded by a constant not relying on* $\varepsilon$ *on a product* $U_{1,d_1} \times H_{\beta'}$ *where* $U_{1,d_1}$ *represents a bounded open sector centered at 0 with bisecting direction* $d_1$, *for any given* $0 < \beta' < \beta$.

(i) *The map* $U_{2,d_1}(u_1, z, \epsilon)$ *solves the Equation (66) for prescribed initial data* $U_{2,d_1}(0, z, \epsilon) \equiv 0$. *The map* $U_{1,d_1}(u_1, z, \epsilon)$ *is subjected to the Equation (67) for vanishing data* $U_{1,d_1}(0, z, \epsilon) \equiv 0$

*The sector* $U_{1,d_1}$ *is submitted to the next technical constraints:*

(1) *A positive real number* $\Delta_1 > 0$ *can be singled out with the next property: for all* $u_1 \in U_{1,d_1}$, *a direction* $d_{1,u_1} \in \mathbb{R}$ *(that might rely on* $u_1$*) can be favoured with*

$$e^{\sqrt{-1}d_{1,u_1}} \in S_{d_1}, \cos\left(k_1\left(d_{1,u_1} - \arg(u_1)\right)\right) > \Delta_1. \tag{210}$$

(2) *The radius* $r_{U_{1,d_1}} > 0$ *of* $U_{1,d_1}$ *withstands the next upper bounds*

$$0 < r_{U_{1,d_1}} < \Delta_1^{1/k_1}\frac{|\epsilon|}{\left(\nu + \tilde{\Delta}_1\right)^{1/k_1}}, \tag{211}$$

*for some positive real number* $\tilde{\Delta}_1 > 0$, *where* $\Delta_1 > 0$ *is defined in the above item.*

*Proof.* We discuss the first item of the proposition. We mind the maps $\omega_{2,d_1}$ and $\omega_{1,d_1}$ constructed in Propositions 14 and 16, and we select a bounded sector $U_{1,d_1}$ that matches the above prerequisite (210) and (211). We set $u_1 \in U_{1,d_1}$ and take

$$\tau = re^{\sqrt{-1}d_{1,u_1}} \in L_{d_{1,u_1}}, \tag{212}$$

for given real number $r \geq 0$ where $d_{1,u_1}$ is the direction chosen above. Then, then next two inequalities for the Borel map hold.

(i) A constant $\varpi_2 > 0$ can be found for which the next bounds

$$\begin{aligned} &\left|\omega_{2,d_1}(\tau, m, \epsilon)\right| \left|\exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right)\right| \left|e^{\sqrt{-1}zm}\right| \left|\frac{1}{\tau}\right| \\ &\leq \varpi_2 (1+|m|)^{-\mu} e^{-\beta|m|} \frac{1}{|\epsilon|} \exp\left(\nu\left(\frac{r}{|\epsilon|}\right)^{k_1}\right) \\ &\quad \cdot \exp\left(-\left(\frac{r}{|u_1|}\right)^{k_1} \cos\left(k_1\left(d_{1,u_1} - \arg(u_1)\right)\right)\right) \\ &\quad \cdot e^{-m \operatorname{Im}(z)} \\ &\leq \varpi_2 (1+|m|)^{-\mu} e^{-(\beta-\beta')|m|} \frac{1}{|\epsilon|} \exp\left(\nu\left(\frac{r}{|\epsilon|}\right)^{k_1}\right) \\ &\quad \cdot \exp\left(-\left(\frac{r}{|u_1|}\right)^{k_1} \Delta_1\right) \\ &\leq \varpi_2 (1+|m|)^{-\mu} e^{-(\beta-\beta')|m|} \frac{1}{|\epsilon|} \exp\left(-\left(\frac{\tilde{\Delta}_1}{|\varepsilon|^{k_1}}\right) r^{k_1}\right), \end{aligned} \tag{213}$$

hold for all $r \geq 0$, all $m \in \mathbb{R}$.

(ii) Similarly, a constant $\varpi_1 > 0$ can be singled out with the bounds

$$\begin{aligned} &\left|\omega_{1,d_1}(\tau, m, \epsilon)\right| \left|\exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right)\right| \left|e^{\sqrt{-1}zm}\right| \left|\frac{1}{\tau}\right| \\ &\leq \varpi_1 (1+|m|)^{-\mu} e^{-(\beta-\beta')|m|} \frac{1}{|\epsilon|} \exp\left(-\left(\frac{\tilde{\Delta}_1}{|\epsilon|^{k_1}}\right) r^{k_1}\right), \end{aligned} \tag{214}$$

provided that $r \geq 0$ and $m \in \mathbb{R}$.

As a result, we reach the next two upper bounds for the maps $U_{j,d_1}$, $j = 1, 2$. Namely,

$$\begin{aligned} \left|U_{2,d_1}(u_1, z, \epsilon)\right| &\leq \frac{k_1 \varpi_2}{(2\pi)^{1/2}} \int_0^{+\infty} \frac{1}{|\epsilon|} \exp\left(-\left(\frac{\tilde{\Delta}_1}{|\epsilon|^{k_1}}\right) r^{k_1}\right) \\ &\quad \cdot dr \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \\ &\leq \frac{k_1 \varpi_2}{(2\pi)^{1/2}} \int_0^{+\infty} \exp\left(-\tilde{\Delta}_1 r_1^{k_1}\right) dr_1 \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm, \end{aligned} \tag{215}$$

by means of the change of variable $r = |\epsilon| r_1$ in the integral together with

$$\begin{aligned} \left|U_{1,d_1}(u_1, z, \epsilon)\right| &\leq \frac{k_1 \varpi_1}{(2\pi)^{1/2}} \int_0^{+\infty} \exp\left(-\tilde{\Delta}_1 r_1^{k_1}\right) dr_1 \\ &\quad \cdot \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm, \end{aligned} \tag{216}$$

for all $u_1 \in U_{1,d_1}$, $z \in H_{\beta'}$ and all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$. We observe that the right handside of both (215) and (216) are unconstrained constants relatively to $\epsilon$ on $D_{\epsilon_0} \setminus \{0\}$. The first item ensues.

Concerning the second item, we remind from Proposition 14 (resp., Proposition 16) that the Borel map $\omega_{2,d_1}(\tau, m, \epsilon)$ (resp., $\omega_{1,d_1}(\tau, m, \epsilon)$) is shown to solve the associated convolution Equation (83) (resp., (84) and (85)). By tracking reversedly the computations made in Subsection 3.2, we deduce that for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the next properties hold.

The holomorphic map $U_{2,d_1}(u_1, z, \epsilon)$ given by the expression (209) for $j = 2$ obeys the Equation (81), then fulfills (76) and finally solves (66) on the domain $U_{1,d_1} \times H_{\beta'}$, for prescribed initial data $U_{2,d_1}(0, z, \epsilon) \equiv 0$.

The holomorphic map $U_{1,d_1}(u_1, z, \epsilon)$ expressed in the form (209) for $j = 1$ conforms to the Equation (82), then satisfies (77) and finally is subjected (67) on the domain $U_{1,d_1} \times H_{\beta'}$, for vanishing initial data $U_{1,d_1}(0, z, \epsilon) \equiv 0$.

The second item of Proposition 17 follows. □

*7.2. Construction of a Finite Family of Genuine Solutions to the Coupling (66) and (67) and Sharp Bounds for the Neighboring Differences of Related Maps.* We need to refer to the usual definition of a good covering in $\mathbb{C}^*$ given in the textbook [37].

*Definition 18.* Let $\varsigma \geq 2$ be an integer. We consider a set $\underline{\mathscr{E}} = \{\mathscr{E}_p\}_{0 \leq p \leq \varsigma - 1}$ of open bounded sectors $\mathscr{E}_p$ centered at 0 endowed with the next three properties

(1) The intersection of two neighboring sectors $\mathscr{E}_p$ and $\mathscr{E}_{p+1}$ is not empty for any $0 \leq p \leq \varsigma - 1$, where the convention $\mathscr{E}_\varsigma = \mathscr{E}_0$ is chosen

(2) The intersection of any three sectors $\mathscr{E}_p$, $\mathscr{E}_q$, and $\mathscr{E}_r$ for distinct integers $p, q, r \in \{0, \cdots, \varsigma - 1\}$ is empty

(3) The union of all the sectors $\mathscr{E}_p$ is subjected to

$$\bigcup_{p=0}^{\varsigma-1} \mathscr{E}_p = \frac{U}{\{0\}}, \tag{217}$$

for some neighborhood $U$ of 0 in $\mathbb{C}$.

Such a set $\underline{\mathscr{E}}$ is designated as a good covering in $\mathbb{C}^*$.

The next definition displays some domains in $\mathbb{C}$ which are crucially involved in the set up of genuine solutions.

*Definition 19.* We consider two finite sets of bounded open sectors centered at 0,

$$\underline{\mathcal{U}}_1 = \left\{U_{1,d_p}\right\}_{0\le p\le \varsigma-1}, \underline{\mathcal{E}} = \left\{\mathcal{E}_p\right\}_{0\le p\le\varsigma-1} \tag{218}$$

and a bounded sector $\mathcal{T}$ centered at 0, for which the next list of constraints is required.

(1) For each $0 \le p \le \varsigma - 1$ and fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, for some given radius $\epsilon_0 > 0$, the sector $U_{1,d_p}$ has bisecting direction $d_p \in \mathbb{R}$ and obeys the next three rules

(i) For each $0 \le p \le \varsigma - 1$, one can single out an unbounded sector $S_{d_p}$ centered at 0 with bisecting direction $d_p$ that is subjected to the requirements of Lemma 12 (namely, for which the lower bounds (131) and (132) hold).

(ii) For each $0 \le p \le \varsigma - 1$, a positive real number $\Delta_p > 0$ can be selected in a way that for all $u_1 \in U_{1,d_p}$, a direction $d_{p,u_1}$ (that might depend on $u_1$) can be found with

$$e^{\sqrt{-1}d_{p,u_1}} \in S_{d_p}, \quad \cos\left(k_1\left(d_{p,u_1} - \arg\left(u_1\right)\right)\right) > \Delta_p. \tag{219}$$

(iii) The radius $r_{U_{1,d_p}} > 0$ of $U_{1,d_p}$ is constrained to the next upper bounds

$$0 < r_{U_{1,d_p}} < \Delta_p^{1/k_1} \frac{|\epsilon|}{\left(\nu + \tilde{\Delta}_p\right)^{1/k_1}}, \tag{220}$$

for some positive real number $\tilde{\Delta}_p > 0$, where $\Delta_p > 0$ is determined in the above item.

(2) The radius $r_{\mathcal{T}} > 0$ of the sector $\mathcal{T}$ satisfies the restriction

$$r_{\mathcal{T}} < \frac{\Delta_p^{1/k_1}}{\left(\nu + \tilde{\Delta}_p\right)^{1/k_1}}, \tag{221}$$

where $\Delta_p$, $\tilde{\Delta}_p$ are specified in 1. for $0 \le p \le \varsigma - 1$. Besides, the sectors $\mathcal{E}_p$ share a common radius given by $\epsilon_0$, for $0 \le p \le \varsigma - 1$.

(3) For all $0 \le p \le \varsigma - 1$, the sectors $\mathcal{E}_p$ and $\mathcal{T}$ stick to the feature

$$\epsilon t \in U_{1,d_p} \tag{222}$$

provided that $\epsilon \in \mathcal{E}_p$ and $t \in \mathcal{T}$.

(4) The set $\underline{\mathcal{E}}$ stands for a good covering in $\mathbb{C}^*$. Furthermore, the aperture of the sector $\mathcal{T}$ is taken nearby 0 in a way that the set

$$I_1 = \left\{p \in \frac{\{0, \cdots, \varsigma - 1\}}{\epsilon t} \notin (-\infty, 0], \text{for all } \epsilon \in \mathcal{E}_p, \text{all } t \in \mathcal{T}\right\}, \tag{223}$$

is not empty.

These sets $\underline{\mathcal{U}}_1$ and $\underline{\mathcal{E}}$ and the sector $\mathcal{T}$ form a so-called fitting collection of sectors.

In the next proposition, we shape a finite family of analytic solutions to the coupling of auxiliary problems (66) and (67).

**Proposition 20.** *We consider a fitting collection of sectors $\underline{\mathcal{U}}_1$, $\underline{\mathcal{E}}$, and $\mathcal{T}$ in the sense of Definition 19. The solutions to (66) and (67) are cooked up as follows.*

*Equation (66) possesses a finite set of holomorphic solutions $(u_1, z) \mapsto U_{2,d_p}(u_1, z, \epsilon)$, for $0 \le p \le \varsigma - 1$, on the domain $U_{1,d_p} \times H_{\beta'}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, where $\epsilon_0$ is proximate to 0, for any $0 < \beta' < \beta$, that fulfills the initial condition $U_{2,d_p}(0, z, \epsilon) \equiv 0$. These maps enjoy the next two qualities: for each $0 \le p \le \varsigma - 1$,*

*(1) The map $(u_1, z) \mapsto U_{2,d_p}(u_1, z, \epsilon)$ is bounded by a constant unconstrained to $\epsilon$ in $D_{\epsilon_0} \setminus \{0\}$, on the product $U_{1,d_p} \times H_{\beta'}$.*

*(2) The map $U_{2,d_p}(u_1, z, \epsilon)$ is represented as Fourier inverse and Laplace transforms*

$$U_{2,d_p}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p,u_1}}} \int_{-\infty}^{+\infty} \omega_{2,d_p}(\tau, m, \epsilon) \cdot \exp\left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm, \tag{224}$$

*where the Borel maps $(\tau, m) \mapsto \omega_{2,d_p}(\tau, m, \epsilon)$ appertain to the Banach space.*

$F^{d_p}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ *are subjected to*

$$\sup_{\epsilon \in D_{\epsilon_0} \setminus \{0\}} \left\| \omega_{2,d_p}(\tau, m, \epsilon) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \varpi_2, \tag{225}$$

*for suitable constants $\varpi_2 > 0$ and radius $\rho > 0$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.*

*Equation (67) (where the expression $U_2(u_1, z, \epsilon)$ needs to be replaced by $U_{2,d_p}(u_1, z, \epsilon)$) owns a finite set of holomorphic solutions $(u_1, z) \mapsto U_{1,d_p}(u_1, z, \epsilon)$, for $0 \le p \le \varsigma - 1$, on the domain $U_{1,d_p} \times H_{\beta'}$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, where $\epsilon_0$ is closed* *to 0, for any $0 < \beta' < \beta$, with the initial condition $U_{1,d_p}(0, z, \epsilon) \equiv 0$. These maps are endowed with the next two features: for each $0 \le p \le \varsigma - 1$,*

(1) *The map $(u_1, z) \mapsto U_{1,d_p}(u_1, z, \epsilon)$ is bounded on the product $U_{1,d_p} \times H_{\beta'}$ by a constant not relying to $\epsilon$ in $D_{\epsilon_0} \setminus \{0\}$.*

(2) *The map $U_{1,d_p}(u_1, z, \epsilon)$ is expressed by means of a Fourier inverse and Laplace transforms*

$$U_{1,d_p}(u_1, z, \epsilon) = \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_p,u_1}} \int_{-\infty}^{+\infty} \omega_{1,d_p}(\tau, m, \epsilon) \exp \cdot \left(-\left(\frac{\tau}{u_1}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm, \tag{226}$$

*where the Borel maps $(\tau, m) \mapsto \omega_{1,d_p}(\tau, m, \epsilon)$ are crafted in the Banach space.*

*$F^{d_p}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ with bounds*

$$\sup_{\epsilon \in D_{\epsilon_0} \setminus \{0\}} \left\| \omega_{1,d_p}(\tau, m, \epsilon) \right\|_{(\nu,\beta,\mu,k_1,\rho,\epsilon)} \le \varpi_1, \tag{227}$$

*for appropriate constants $\varpi_1 > 0$ and radius $\rho > 0$, for all $\epsilon \in D_{\epsilon_0} \setminus \{0\}$.*

*Proof.* Proposition 20 is a downright consequence of Proposition 17 and of the very definition of fitting collections of sectors depicted in Definition 19. □

In the next proposition, we examine a finite set of maps related to the analytic solutions of the coupling (66) and (67). In particular, we obtain a control on their consecutive differences which turns out to be an essential information in the study of their parametric asymptotic expansions.

**Proposition 21.** *Let us prescribe a fitting collection of sectors $\underline{\mathcal{U}_1}$, $\underline{\mathcal{E}}$, and $\mathcal{T}$ in accordance with Definition 19. For each $0 \le p \le \varsigma - 1$, we set up the maps*

$$u_{j,p}(t, z, \epsilon) = U_{j,d_p}(\epsilon t, z, \epsilon), \tag{228}$$

*for $j = 1, 2$, where $U_{j,d_p}$ are described in Proposition 20. The next attributes hold: for all $0 \le p \le \varsigma - 1$,*

(i) *The maps $u_{j,p}(t, z, \epsilon)$, $j = 1, 2$, are bounded holomorphic on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ and satisfy $u_{j,p}(0, z, \epsilon) \equiv 0$,*

(ii) *One can exhibit constants $M_{p,j} > 0$ and $K_{p,j} > 0$ such that*

$$\left| u_{j,p+1}(t, z, \epsilon) - u_{j,p}(t, z, \epsilon) \right| \le M_{p,j} \exp\left(-\frac{K_{p,j}}{|\epsilon|^{k_1}}\right), \tag{229}$$

*for all $t \in \mathcal{T}$, all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, all $z \in H_{\beta'}$, for $j = 1, 2$, where we adopt the convention $u_{j,\varsigma} = u_{j,0}$.*

*Proof.* The first item is a direct outcome of the properties of the maps $U_{j,d_p}$, $j = 1, 2$, stated in Proposition 20 and from the characteristics 2 and 3 of the sectors $\mathcal{E}_p$ and $\mathcal{T}$ listed in Definition 19.

The second item follows from a path deformation argument. Indeed, let us take $p \in \{0, \cdots, \varsigma - 1\}$ and $j \in \{1, 2\}$. For any given $m \in \mathbb{R}$ and fixed $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, the partial maps $\tau \mapsto \omega_{j,d_k}(\tau, m, \epsilon)$, $k = p, p + 1$, represent analytic continuation on the sector $S_{d_k}$ of a common analytic map we denote $\tau \mapsto \omega_j(\tau, m, \epsilon)$ on the disc $D_\rho$.

For any prescribed $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$ and $t \in \mathcal{T}$, we deform the oriented path $L_{d_{p+1,\epsilon t}} - L_{d_{p,\epsilon t}}$ into the union of three oriented curves

(i) Two halflines

$$\begin{aligned} L_{d_{p+1,\epsilon t};\rho/2} &= [\rho/2,+\infty) e^{\sqrt{-1} d_{p+1,\epsilon t}}, -L_{d_{p,\epsilon t};\rho/2} \\ &= -[\rho/2,+\infty) e^{\sqrt{-1} d_{p,\epsilon t}}. \end{aligned} \tag{230}$$

(ii) An arc of circle

$$C_{p,p+1,\epsilon t;\rho/2} = \left\{ \frac{\rho/2 e^{\sqrt{-1}\theta}}{\theta} \in \left(d_{p,\epsilon t}, d_{p+1,\epsilon t}\right) \right\}, \tag{231}$$

centered at 0 with radius $\rho/2$ that connects the above two halflines.

Then, the classical Cauchy's theorem enables us to reshape the next difference into a sum of three contributions

$$\begin{aligned} & u_{j,p+1}(t, z, \epsilon) - u_{j,p}(t, z, \epsilon) \\ &= \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p+1,\epsilon t};\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_{p+1}}(\tau, m, \epsilon) \exp\left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) \\ &\quad \cdot e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm - \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p,\epsilon t};\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_p}(\tau, m, \epsilon) \exp \\ &\quad \cdot \left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm + \frac{k_1}{(2\pi)^{1/2}} \int_{C_{p,p+1,\epsilon t;\rho/2}} \\ &\quad \cdot \int_{-\infty}^{+\infty} \omega_j(\tau, m, \epsilon) \exp\left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm. \end{aligned} \tag{232}$$

We provide upper bounds for the first piece of (232)

$$I_1 = \left| \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_{p+1},\epsilon t;\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_{p+1}}(\tau, m, \epsilon) \cdot \exp\left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \right|. \tag{233}$$

Based on the bounds (213)–(225) and (227) together with the requirements asked in Definition 19, we arrive at

$$\begin{aligned} I_1 &\le \frac{\varpi_j k_1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \times \int_{\rho/2}^{+\infty} \frac{1}{|\epsilon|} \exp\left(-\frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} r^{k_1}\right) dr \\ &\le \frac{2\varpi_j k_1}{(2\pi)^{1/2}} \int_{0}^{+\infty} e^{-(\beta-\beta')m} dm \times \int_{\rho/2}^{+\infty} \frac{1}{|\epsilon|} \left\{ \frac{|\epsilon|^{k_1}}{\tilde{\Delta}_{p+1}} \frac{1}{k_1 r^{k_1-1}} \right\} \\ &\quad \cdot \left\{ \frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} k_1 r^{k_1-1} \exp\left(-\frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} r^{k_1}\right) \right\} dr \\ &\le \frac{2\varpi_j k_1}{(2\pi)^{1/2}} \frac{1}{\beta-\beta'} \frac{|\epsilon|^{k_1-1}}{\tilde{\Delta}_{p+1}} \frac{1}{k_1 (\rho/2)^{k_1-1}} \exp\left(-\frac{\tilde{\Delta}_{p+1}}{|\epsilon|^{k_1}} \left(\frac{\rho}{2}\right)^{k_1}\right), \end{aligned} \tag{234}$$

provided that $\epsilon \in \mathscr{E}_{p+1} \cap \mathscr{E}_p$, $t \in \mathscr{T}$ and $z \in H_{\beta'}$.

In the same vein, we can get upper bounds for the second piece

$$I_2 = \left| \frac{k_1}{(2\pi)^{1/2}} \int_{L_{d_p,\epsilon t;\rho/2}} \int_{-\infty}^{+\infty} \omega_{j,d_p}(\tau, m, \epsilon) \cdot \exp\left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \right|, \tag{235}$$

of (232). Namely,

$$|I_2| \le \frac{2\varpi_j k_1}{(2\pi)^{1/2}} \frac{1}{\beta-\beta'} \frac{|\epsilon|^{k_1-1}}{\tilde{\Delta}_p} \frac{1}{k_1 (\rho/2)^{k_1-1}} \exp\left(-\frac{\tilde{\Delta}_p}{|\epsilon|^{k_1}} \left(\frac{\rho}{2}\right)^{k_1}\right), \tag{236}$$

for all $\epsilon \in \mathscr{E}_{p+1} \cap \mathscr{E}_p$, $t \in \mathscr{T}$ and $z \in H_{\beta'}$.

At last, we handle the integral along the arc of circle closing (232),

$$I_3 = \left| \frac{k_1}{(2\pi)^{1/2}} \int_{C_{p,p+1,\epsilon t;\rho/2}} \int_{-\infty}^{+\infty} \omega_j(\tau, m, \epsilon) \exp \cdot \left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right) e^{\sqrt{-1}zm} \frac{d\tau}{\tau} dm \right|. \tag{237}$$

Owing to the bounds (225) and (227), we observe that

$$\left|\omega_j(\tau, m, \epsilon)\right| \le \varpi_j (1+|m|)^{-\mu} e^{-\beta|m|} \frac{\rho/2}{|\epsilon|} \exp\left(\nu \frac{(\rho/2)^{k_1}}{|\epsilon|^{k_1}}\right), \tag{238}$$

as long as $\tau \in C_{p,p+1,\epsilon t;\rho/2}$, $m \in \mathbb{R}$ and $\epsilon \in \mathscr{E}_{p+1} \cap \mathscr{E}_p$. Besides, in view of the restrictions discussed in Definition 6.1, it follows that

$$\cos\left(k_1(\theta - \arg(\epsilon t))\right) > \Delta_{p,p+1} = \min\left(\Delta_p, \Delta_{p+1}\right), \tag{239}$$

for all $t \in \mathscr{T}$, $\epsilon \in \mathscr{E}_{p+1} \cap \mathscr{E}_p$, granted that the angle $\theta$ belongs to $(d_{p,\epsilon t}, d_{p+1,\epsilon t})$ or $(d_{p+1,\epsilon t}, d_{p,\epsilon t})$. By virtue of (238) and (239), we come up with some constant $\tilde{\Delta}_{p,p+1} > 0$ with

$$\begin{aligned} I_3 &\le \frac{k_1 \varpi_j}{(2\pi)^{1/2}} \left( \int_{-\infty}^{+\infty} e^{-(\beta-\beta')|m|} dm \right) \\ &\quad \times \left| \int_{d_{p,\epsilon t}}^{d_{p+1,\epsilon t}} \frac{1}{|\epsilon|} \exp\left(\nu' \frac{(\rho/2)^{k_1}}{|\epsilon|^{k_1}}\right) \exp\left(-\frac{(\rho/2)^{k_1}}{|\epsilon t|^{k_1}} \Delta_{p,p+1}\right) \frac{\rho}{2} d\theta \right| \\ &\le \frac{2k_1 \varpi_j}{(2\pi)^{1/2}(\beta-\beta')} \left|d_{p+1,\epsilon t} - d_{p,\epsilon t}\right| \frac{1}{|\epsilon|} \exp\left(-\frac{\tilde{\Delta}_{p,p+1}}{|\epsilon|^{k_1}} \left(\frac{\rho}{2}\right)^{k_1}\right) \frac{\rho}{2}, \end{aligned} \tag{240}$$

contingent upon $t \in \mathscr{T}$, $\epsilon \in \mathscr{E}_{p+1} \cap \mathscr{E}_p$, and $z \in H_{\beta'}$. Hence, we deduce that

$$\begin{aligned} I_3 &\le \frac{2k_1 \varpi_j}{(2\pi)^{1/2}(\beta-\beta')} \left|d_{p+1,\epsilon t} - d_{p,\epsilon t}\right| \frac{\rho}{2} \frac{1}{|\epsilon|} \\ &\quad \cdot \exp\left(-\frac{\tilde{\Delta}_{p,p+1}}{2|\epsilon|^{k_1}} \left(\frac{\rho}{2}\right)^{k_1}\right) \exp\left(-\frac{\tilde{\Delta}_{p,p+1}}{2|\epsilon|^{k_1}} \left(\frac{\rho}{2}\right)^{k_1}\right) \\ &\le \frac{2k_1 \varpi_j}{(2\pi)^{1/2}(\beta-\beta')} \left|d_{p+1,\epsilon t} - d_{p,\epsilon t}\right| \frac{\rho}{2} \mathscr{C}_{k_1,\rho,\tilde{\Delta}_{p,p+1}} \\ &\quad \cdot \exp\left(-\frac{\tilde{\Delta}_{p,p+1}}{2|\epsilon|^{k_1}} \left(\frac{\rho}{2}\right)^{k_1}\right), \end{aligned} \tag{241}$$

holds, where

$$\mathscr{C}_{k_1,\rho,\tilde{\Delta}_{p,p+1}} = \sup_{x \ge 0} x \exp\left(-\frac{\tilde{\Delta}_{p,p+1}}{2} \left(\frac{\rho}{2}\right)^{k_1} x^{k_1}\right), \tag{242}$$

as long as $\epsilon \in \mathscr{E}_{p+1} \cap \mathscr{E}_p$, $t \in \mathscr{T}$ and $z \in H_{\beta'}$.

In summary, the splitting (232) along with the bounds (234), (236), and (241) beget the awaited estimates (229). □

## 8. Main Statement of the Paper: Construction of a Finite Set of Holomorphic Solutions to the Leading Problem (37)—Description of their Parametric Asymptotic Expansion

*8.1. Parametric Gevrey Asymptotic Expansions of the Associated Maps (228).* We first call to mind a result known as the Ramis-Sibuya theorem, see Lemma XI-2-6 in [37].

**Theorem 22.** *Let $\{\mathcal{E}_p\}_{0\le p\le \varsigma-1}$ be a good covering in $\mathbb{C}^*$ be fixed as described in Definition 18. We denote $(\mathbb{F}, \|.\|_{\mathbb{F}})$ a Banach space over $\mathbb{C}$. For all $0\le p\le \varsigma-1$, we set $G_p : \mathcal{E}_p \longrightarrow \mathbb{F}$ as holomorphic functions that obey the next requirements*

*(1) The maps $G_p$ are bounded on $\mathcal{E}_p$ for all $0\le p\le \varsigma-1$*

*(2) The difference $\Theta_p(\varepsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ defines a holomorphic map on the intersection $Z_p = \mathcal{E}_{p+1}\cap\mathcal{E}_p$ which is exponentially flat of order $k_1$, for some integer $k_1\ge 1$, meaning that one can select two constants $C_p, A_p > 0$ for which*

$$\left\|\Theta_p(\epsilon)\right\|_{\mathbb{F}} \le C_p \exp\left(-\frac{A_p}{|\epsilon|^{k_1}}\right), \tag{243}$$

*holds provided that $\epsilon\in Z_p$, for all $0\le p\le \varsigma-1$. By convention, we set $G_\varsigma = G_0$ and $\mathcal{E}_\varsigma = \mathcal{E}_0$.*

*Then, one can find a formal power series $\widehat{G}(\epsilon) = \sum_{n\ge 0} G_n\epsilon^n$ with coefficients $G_n$ belonging to $\mathbb{F}$, which is the common Gevrey asymptotic expansion of order $1/k_1$ relatively to $\epsilon$ on $\mathcal{E}_p$ for all the maps $G_p$, for $0\le p\le \varsigma-1$. It means that two constants $K_p, M_p > 0$ can be singled out with the error bounds*

$$\left\|G_p(\epsilon) - \sum_{n=0}^{N} G_n\epsilon^n\right\|_{\mathbb{F}} \le K_p M_p^{N+1}\Gamma\left(1+\frac{N+1}{k_1}\right)|\epsilon|^{N+1}, \tag{244}$$

*for all integers $N\ge 0$, all $\varepsilon\in\mathcal{E}_p$, all $0\le p\le \varsigma-1$.*

In the next proposition, we exhibit asymptotic expansions of Gevrey type for the two sets of related maps introduced in Proposition 21, $\{u_{j,p}(t,z,\epsilon)\}_{0\le p\le\varsigma-1}$, $j=1,2$, relatively to the variable $\epsilon$.

**Proposition 23.** *We denote $\mathbb{F}_{\beta',\mathcal{T}}$ the Banach space of bounded holomorphic functions on the product $\mathcal{T}\times H_{\beta'}$ which are $\mathbb{C}$ – valued, equipped with the sup norm. Then, for $j=1,2$, a formal power series*

$$\widehat{\mathbb{G}}_j(\epsilon) = \sum_{n\ge 0}\mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}, \tag{245}$$

*with coefficients $\mathbb{G}_{n,j}(t,z)$, $n\ge 0$, in $\mathbb{F}_{\beta',\mathcal{T}}$ can be shaped that obey the next error bounds. For all $0\le p\le\varsigma-1$, two constants $K_{p,j}>0$ and $M_{p,j}>0$ can be chosen with*

$$\begin{aligned}\sup_{\substack{t\in\mathcal{T}\\ z\in H_{\beta'}}}&\left|u_{j,p}(t,z,\epsilon) - \sum_{n=0}^{N}\mathbb{G}_{n,j}(t,z)\frac{\epsilon^n}{n!}\right|\\ &\le K_{p,j}\left(M_{p,j}\right)^{N+1}\Gamma\left(1+\frac{N+1}{k_1}\right)|\epsilon|^{N+1},\end{aligned} \tag{246}$$

*for all integers $N\ge 0$, all $\epsilon\in\mathcal{E}_p$.*

*Proof.* Let $j=1,2$. For all $0\le p\le\varsigma-1$, let us define the maps $G_{j,p} : \mathcal{E}_p \longrightarrow \mathbb{F}_{\beta',\mathcal{T}}$ by the expression $G_{j,p}(\epsilon) := (t,z)\mapsto u_{j,p}(t,z,\epsilon)$. For $0\le p\le\varsigma-1$, these functions share the next two features:

(i) The maps $G_{j,p}$ are bounded holomorphic on the sector $\mathcal{E}_p$, according to the first item of Proposition 21

(ii) The differences $\Theta_{j,p}(\epsilon) := G_{j,p+1}(\epsilon) - G_{j,p}(\epsilon)$ are submitted to the bounds

$$\left\|\Theta_{j,p}(\epsilon)\right\|_{\mathbb{F}_{\beta',\mathcal{T}}} \le M_{p,j}\exp\left(-\frac{K_{p,j}}{|\epsilon|^{k_1}}\right), \tag{247}$$

for the constants $M_{p,j}>0$ and $K_{p,j}>0$ obtained in Proposition 21, whenever $\epsilon\in\mathcal{E}_{p+1}\cap\mathcal{E}_p$, where the convention $G_{j,\varsigma} = G_{j,0}$ and $\mathcal{E}_\varsigma = \mathcal{E}_0$ is in use.

As a result, requirements 1 and 2 of Theorem (22) are matched for the sets of maps $\{G_{j,p}\}_{0\le p\le\varsigma-1}$, $j=1,2$. We deduce the existence of formal series $\widehat{\mathbb{G}}_j(\epsilon)$, $j=1,2$, which are the Gevrey asymptotic expansion of order $1/k_1$ relatively to $\epsilon$ on $\mathcal{E}_p$ shared by all the maps $G_{j,p}$ for $0\le p\le\varsigma-1$. This is tantamount to the statement of Proposition 23 and the awaited bounds (246). □

*8.2. Statement of the Main Result.* The next statement stands for the pinnacle of our work.

**Theorem 24.** *Let us prescribe a fitting collection of sectors $\underline{\mathcal{U}}_1$, $\underline{\mathcal{E}}$, and $\mathcal{T}$ accordingly to Definition 19. We take for granted that all the conditions (38)–(45), (47)–(49), (54), and (55) enumerated in Subsection 2.2 are fulfilled.*

*Then, provided that the constants $\epsilon_0>0$ and $\mathbf{C}_{1,\epsilon_0}>0$, $\mathbf{B}_{j,\epsilon_0}>0$, $j=1,2$, along with $c_{Q_1Q_2}\in\mathbb{C}^*$ and $c_{P_jP_{j+1}}\in\mathbb{C}^*$, $j=1,3,5$ are nonvanishing and taken proximate to 0, the main equation*

$$\begin{aligned}Q(\partial_z)u(t,z,\epsilon) &= (\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)u(t,z,\epsilon)\\ &+\sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)u(t,z,\epsilon)\end{aligned}$$

$$+f(t,z,\epsilon)+c_1(z,\epsilon)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\cdot\log(\epsilon t)+b_1(z,\epsilon)\cdot\left[u(t,z,\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\log(\epsilon t)\right]+b_2(z,\epsilon)\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)+c_{Q_1Q_2}Q_1(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\times Q_2(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\times\log(\epsilon t)+c_{P_1P_2}P_1(\partial_z)\cdot\left[u(t,z,\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\log(\epsilon t)\right]\times P_2(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]+c_{P_3P_4}P_3(\partial_z)\cdot\left[u(t,z,\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\log(\epsilon t)\right]\times P_4(\partial_z)\left[u(t,z,\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\log(\epsilon t)\right]+c_{P_5P_6}P_5(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right]\times P_6(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)u(t,z,\epsilon)\right], \tag{248}$$

*with vanishing initial data*

$$u(0,z,\epsilon)\equiv 0 \tag{249}$$

*possesses a finite set of bounded holomorphic solutions $(t,z,\epsilon)\mapsto u_p(t,z,\epsilon)$, for all $p\in I_1$, where $I_1$ is the subset of $\{0,\cdots,\varsigma-1\}$ introduced in item 4 of Definition 19, on the domain $\mathcal{T}\times H_{\beta'}\times\mathcal{E}_p$. In Equation (248), the formal monodromy operator around 0, $\gamma_\epsilon^*$ acts on the analytic map $\epsilon\mapsto u_p(t,z,\epsilon)$ through Definition 2 by use of (34). The next additional features hold.*

(i) *For each $p\in I_1$, the solution $u_p$ can be expressed by means of a Fourier/Laplace transform*

$$u_p(t,z,\epsilon)=u_{1,p}(t,z,\epsilon)+u_{2,p}(t,z,\epsilon)\log(\epsilon t), \tag{250}$$

*where*

$$u_{j,p}(t,z,\epsilon)=\frac{k_1}{(2\pi)^{1/2}}\int_{L_{d_p,\epsilon t;\rho/2}}\int_{-\infty}^{+\infty}\omega_{j,d_p}(\tau,m,\epsilon)\cdot\exp\left(-\left(\frac{\tau}{\epsilon t}\right)^{k_1}\right)e^{\sqrt{-1}zm}\frac{d\tau}{\tau}dm, \tag{251}$$

*for Borel maps $(\tau,m)\mapsto\omega_{j,d_p}(\tau,m,\epsilon)$, $j=1,2$, that belong to the Banach space $F^{d_p}_{(\nu,\beta,\mu,k_1,\rho,\epsilon)}$ under restrictions (194) and (196).*

(ii) *The two components $u_{j,p}(t,z,\epsilon)$, $j=1,2$, of $u_p(t,z,\epsilon)$ are endowed with Gevrey asymptotic expansions $\widehat{\mathbb{G}}_j(\epsilon)$ given by (245) of order $1/k_1$ relatively to $\epsilon$ on $\mathcal{E}_p$ displayed in (246).*

(iii) *If one sets the formal expression*

$$\widehat{\mathbb{G}}(\epsilon)=\widehat{\mathbb{G}}_1(\epsilon)+\widehat{\mathbb{G}}_2(\epsilon)\log(\epsilon t), \tag{252}$$

*then, $\widehat{\mathbb{G}}(\epsilon)$ conforms to the next equation*

$$Q(\partial_z)\widehat{\mathbb{G}}(\epsilon)=(\epsilon t)^{d_D}(t\partial_t)^{\delta_D}R_D(\partial_z)\widehat{\mathbb{G}}(\epsilon)+\sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)\widehat{\mathbb{G}}(\epsilon)+f(t,z,\epsilon)+c_1(z,\epsilon)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\log(\epsilon t)+b_1(z,\epsilon)\left[\widehat{\mathbb{G}}(\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\cdot\log(\epsilon t)\right]+b_2(z,\epsilon)\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)+c_{Q_1Q_2}Q_1(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\times Q_2(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\times\log(\epsilon t)+c_{P_1P_2}P_1(\partial_z)\left[\widehat{\mathbb{G}}(\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\cdot\log(\epsilon t)\right]\times P_2(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]+c_{P_3P_4}P_3(\partial_z)\left[\widehat{\mathbb{G}}(\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\cdot\log(\epsilon t)\right]\times P_4(\partial_z)\left[\widehat{\mathbb{G}}(\epsilon)-\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\log(\epsilon t)\right]+c_{P_5P_6}P_5(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right]\times P_6(\partial_z)\left[\frac{1}{2\sqrt{-1}\pi}(\gamma_\epsilon^*-id)\widehat{\mathbb{G}}(\epsilon)\right], \tag{253}$$

*where the formal monodromy operator around 0, $\gamma_\epsilon^*$ acts on the formal expression $\epsilon\mapsto\widehat{\mathbb{G}}(\epsilon)$ by means of the formula (32) from Definition 1.*

*Proof.* For all $p \in I_1$, where $I_1$ is the set described in item 4 of Definition 19, we define

$$u_p(t,z,\epsilon) = u_{1,p}(t,z,\epsilon) + u_{2,p}(t,z,\epsilon)\log(\epsilon t), \tag{254}$$

where the maps $u_{j,p}$ are introduced in (228) of Proposition 21.

As a result of the definition of $I_1$ together with the first item of Proposition 21 and the classical limit $\lim_{x\longrightarrow 0} x^{\alpha}\log(x) = 0$, for any natural number $\alpha \geq 1$, we check that the map $u_p(t,z,\epsilon)$ represents a bounded holomorphic function on the product $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$ that vanishes at $t = 0$, meaning that $u_p(0,z,\epsilon) \equiv 0$ for all $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$.

According to Proposition 20, we know that for each $\epsilon \in D_{\epsilon_0} \setminus \{0\}$,the map $(u_1,z) \mapsto U_{2,d_p}(u_1,z,\epsilon)$ stands for a solution of Equation (66) on the domain $U_{1,d_p} \times H_{\beta'}$.

The map $(u_1,z) \mapsto U_{1,d_p}(u_1,z,\epsilon)$ embodies a solution of (67) where the expression $U_2(u_1,z,\epsilon)$ is asked to be replaced by $U_{2,d_p}(u_1,z,\epsilon)$ on the domain $U_{1,d_p} \times H_{\beta'}$.

Then, on the basis of the computations (65), (63), and (62) performed reversedly from Subsection 3.1, we deduce that $u_p(t,z,\epsilon)$ solves the main Equation (37), rephrased as (248), on the domain $\mathcal{T} \times H_{\beta'} \times \mathcal{E}_p$, for all $p \in I_1$.

The first item of Theorem 24 follows from the Fourier/Laplace representation of the maps $U_{j,d_p}(u_1,z,\epsilon)$, $j = 1,2$, displayed in Proposition 20 that are used to define the components $u_{j,p}(t,z,\epsilon)$ in (197).

The second item of Theorem 24 merely restates the result obtained in Proposition 23.

We focus on the third item. We first need to disclose partial differential equations that the maps $u_{j,p}(t,z,\epsilon)$, $j = 1,2$ turn out to fulfill. Indeed, the usual chain rule enables the next computation

$$t\partial_t u_{j,p}(t,z,\epsilon) = \left(u_1 \partial_{u_1} U_{j,d_p}\right)(\epsilon t, z, \epsilon), \tag{255}$$

for all $0 \leq p \leq \varsigma - 1$, $j = 1,2$, provided that $t \in \mathcal{T}$, $\epsilon \in \mathcal{E}_p$ and $z \in H_{\beta'}$. According to the statement discussed in Proposition 20, the partial map $(u_1,z) \mapsto U_{2,d_p}(u_1,z,\epsilon)$ matches Equation (66) on the domain $U_{1,d_p} \times H_{\beta'}$, whenever $\epsilon \in D_{\epsilon_0} \setminus \{0\}$; we observe that the map $u_{2,p}(t,z,\epsilon)$ satisfies the next equation

$$\begin{aligned} Q(\partial_z)u_{2,p}(t,z,\epsilon) = {} & (\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D} R_D(\partial_z) u_{2,p}(t,z,\epsilon)\right] \\ & + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)(t\partial_t)^{\delta_l} R_l(\partial_z) u_{2,p}(t,z,\epsilon) \\ & + F_2(\epsilon t,z,\epsilon) + c_1(z,\epsilon) u_{2,p}(t,z,\epsilon) \\ & + c_{Q_1,Q_2}\left[Q_1(\partial_z)u_{2,p}(t,z,\epsilon))\right] \\ & \times \left[Q_2(\partial_z)u_{2,p}(t,z,\epsilon)\right], \end{aligned} \tag{256}$$

as long as $t \in \mathcal{T}$, $z \in H_{\beta'}$, and $\epsilon \in \mathcal{E}_p$. On the other hand, since the partial map $(u_1,z) \mapsto U_{1,d_p}(u_1,z,\epsilon)$ obeys Equation (67) on the domain $U_{1,d_p} \times H_{\beta'}$, for $\epsilon \in D_{\epsilon_0} \setminus \{0\}$, it follows that the map $u_{1,p}(t,z,\epsilon)$ fulfills the next equation coupled to (215),

$$\begin{aligned} Q(\partial_z)u_{1,p}(t,z,\epsilon) = {} & (\epsilon t)^{d_D}\Big[(t\partial_t)^{\delta_D} R_D(\partial_z) u_{1,p}(t,z,\epsilon) \\ & + \delta_D (t\partial_t)^{\delta_D - 1} R_D(\partial_z) u_{2,p}(t,z,\epsilon)\Big] \\ & + \sum_{l=1}^{D-1} \epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)\Big[(t\partial_t)^{\delta_l} R_l(\partial_z) u_{1,p}(t,z,\epsilon) \\ & + \delta_l (t\partial_t)^{\delta_l - 1} R_l(\partial_z) u_{2,p}(t,z,\epsilon)\Big] \\ & + F_1(\epsilon t,z,\epsilon) + b_1(z,\epsilon) u_{1,p}(t,z,\epsilon) \\ & + b_2(z,\epsilon) u_{2,p}(t,z,\epsilon) \\ & + c_{P_1P_2}\left[P_1(\partial_z)u_{1,p}(t,z,\epsilon)\right] \\ & \times \left[P_2(\partial_z)u_{2,p}(t,z,\epsilon)\right] \\ & + c_{P_3P_4}\left[P_3(\partial_z)u_{1,p}(t,z,\epsilon)\right] \\ & \times \left[P_4(\partial_z)u_{1,p}(t,z,\epsilon)\right] \\ & + c_{P_5P_6}\left[P_5(\partial_z)u_{2,p}(t,z,\epsilon)\right] \\ & \times \left[P_6(\partial_z)u_{2,p}(t,z,\epsilon)\right], \end{aligned} \tag{257}$$

provided that $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$.

The next classical result (stated in Proposition 17 p. 66 from [19]) will be essential to deduce recursion relations for the coefficients $\mathbb{G}_{n,j}(t,z)$, $n \geq 0$ of $\widehat{\mathbb{G}}_j(\epsilon)$ from the partial differential equations that govern the components $u_{j,p}(t,z,\epsilon)$, $j = 1,2$. □

**Proposition 25.** *Let $f : G \longrightarrow \mathbb{F}$ be a holomorphic map from a bounded open sector $G$ centered at 0 in $\mathbb{C}^*$ into a complex Banach space $\mathbb{F}$ equipped with a norm $\|.\|_{\mathbb{F}}$. The next statements are equivalent*

(i) *There exists a formal power series $\widehat{f}(\epsilon) = \sum_{n\geq 0} f_n \epsilon^n/n!$ in $\mathbb{F}[[\epsilon]]$ which is the asymptotic expansion of $f$ on $G$, meaning that for all closed sector $S$ of $G$ centered at 0, one can associate a sequence $(c(N,S))_{N\geq 0}$ of positive real numbers such that*

$$\left\| f(\epsilon) - \sum_{n=0}^{N-1} f_n \epsilon^n/n! \right\|_{\mathbb{F}} \leq c(N,S)|\epsilon|^N, \tag{258}$$

*for all $\epsilon \in S$, all integers $N \geq 1$.*

(ii) *All $n-th$ derivatives of $f$ denoted $f^{(n)}(\epsilon)$ are continuous at 0 and satisfy*

$$\lim_{\substack{\epsilon\longrightarrow 0\\ \epsilon\in G}} \left\| f^{(n)}(\epsilon) - f_n \right\|_{\mathbb{F}} = 0, \tag{259}$$

*for all integers $n \geq 0$.*

We first derive some recursion relations for the coefficients $\mathbb{G}_{m,2}(t,z)$, $m \geq 0$. To that aim, we take the derivative of order $m \geq 0$ on the left and right handsides of (256) relatively to $\varepsilon$ for any integer $m \geq 0$. Indeed, owing to the Leibniz rule, we deduce

$$\begin{aligned} Q(\partial_z)\partial_\epsilon^m u_{2,p}(t,z,\epsilon) &= \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\partial_\epsilon^{m_1} \epsilon^{d_D}\right) t^{d_D}(t\partial_t)^{\delta_D} R_D(\partial_z) \\ &\cdot \left[\partial_\epsilon^{m_2} u_{2,p}(t,z,\epsilon)\right] + \sum_{l=1}^{D-1} \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!} \\ &\cdot \left(\partial_\epsilon^{m_1} \epsilon^{\Delta_l}\right) t^{d_l} \times [(\partial_\epsilon^{m_2} a_l(z,\epsilon)] \\ &\times (t\partial_t)^{\delta_l} R_l(\partial_z)\left[\partial_\epsilon^{m_3} u_{2,p}(t,z,\epsilon)\right] + \partial_\epsilon^m F_2(\epsilon t,z,\epsilon) \\ &+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} [\partial_\epsilon^{m_1} c_1(z,\epsilon)] \times \left[\partial_\epsilon^{m_2} u_{2,p}(t,z,\epsilon)\right] \\ &+ c_{Q_1,Q_2} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left[Q_1(\partial_z)\partial_\epsilon^{m_1} u_{2,p}(t,z,\epsilon)\right] \\ &\times \left[Q_2(\partial_z)\partial_\epsilon^{m_2} u_{2,p}(t,z,\epsilon)\right], \end{aligned} \tag{260}$$

for all $m \geq 0$, all $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$. Owing to the asymptotic expansion (246) for $j = 2$, the application of Proposition 25 yields the next limits

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon \in \mathcal{E}_p}} \sup_{\substack{t \in \mathcal{T} \\ z \in H_{\beta'}}} \left|\partial_\epsilon^m u_{2,p}(t,z,\epsilon) - \mathbb{G}_{m,2}(t,z)\right| = 0, \tag{261}$$

for all integers $m \geq 0$ and any given $0 \leq p \leq \varsigma - 1$. We let $\epsilon$ tend to 0 on the sector $\mathcal{E}_p$ in the above equality (260) and with the help of (261) combined with the observation that both maps $u_{2,p}(t,z,\epsilon)$ and $\mathbb{G}_{m,2}(t,z)$ are holomorphic with respect to $(t,z)$ on the product $\mathcal{T} \times H_{\beta'}$, we reach the next relation for the coefficients $\mathbb{G}_{m,2}(t,z)$, $m \geq 0$,

$$\begin{aligned} Q(\partial_z)\mathbb{G}_{m,2}(t,z) &= \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left(\partial_\epsilon^{m_1} \epsilon^{d_D}\right)(0) t^{d_D}(t\partial_t)^{\delta_D} \\ &\cdot R_D(\partial_z)\mathbb{G}_{m_2,2}(t,z) + \sum_{l=1}^{D-1} \sum_{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!} \\ &\cdot \left(\partial_\epsilon^{m_1} \epsilon^{\Delta_l}\right)(0) t^{d_l} \times [(\partial_\epsilon^{m_2} a_l)(z,0)] \\ &\times (t\partial_t)^{\delta_l} R_l(\partial_z)\mathbb{G}_{m_3,2}(t,z) + \partial_\epsilon^m F_2(\epsilon t,z,\epsilon)_{|\epsilon=0} \\ &+ \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} [(\partial_\epsilon^{m_1} c_1)(z,0)] \times \mathbb{G}_{m_2,2}(t,z) \\ &+ c_{Q_1,Q_2} \sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!} \left[Q_1(\partial_z)\mathbb{G}_{m_1,2}(t,z)\right] \\ &\times \left[Q_2(\partial_z)\mathbb{G}_{m_2,2}(t,z)\right], \end{aligned} \tag{262}$$

for all $m \geq 0$, provided that $t \in \mathcal{T}$, $z \in H_{\beta'}$.

This enables us to display some partial differential equations fulfilled by the formal expansion $\widehat{\mathbb{G}}_2(\epsilon)$. Namely, we know that the maps $\epsilon \mapsto \epsilon^{d_D}$, $\epsilon \mapsto \epsilon^{\Delta_l}$, $\epsilon \mapsto a_l(z,\epsilon)$ together with $\epsilon \mapsto F_2(\epsilon t,z,\epsilon)$ are analytic on the disc $D_{\epsilon_0}$. Their convergent Taylor series are expressed as

$$\begin{aligned} \epsilon^{d_D} &= \sum_{m\geq 0} \frac{\left(\partial_\epsilon^m \epsilon^{d_D}\right)(0)}{m!} \epsilon^m, \\ \epsilon^{\Delta_l} &= \sum_{m\geq 0} \frac{\left(\partial_\epsilon^m \epsilon^{\Delta_l}\right)(0)}{m!} \epsilon^m, \\ a_l(z,\epsilon) &= \sum_{m\geq 0} \frac{(\partial_\epsilon^m a_l)(z,0)}{m!} \epsilon^m, \end{aligned} \tag{263}$$

$$\begin{aligned} c_1(z,\epsilon) &= \sum_{m\geq 0} \frac{(\partial_\epsilon^m c_1)(z,0)}{m!} \epsilon^m, \\ F_2(\epsilon t,z,\epsilon) &= \sum_{m\geq 0} \frac{\partial_\epsilon^m F_2(\epsilon t,z,\epsilon)_{|\epsilon=0}}{m!} \epsilon^m, \end{aligned} \tag{264}$$

for all $\epsilon \in D_{\epsilon_0}$. Then, departing from (245), we get the formal Taylor expansion of the next pieces that involve $\widehat{\mathbb{G}}_2(\epsilon)$. Namely,

$$\begin{aligned} &(\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D} R_D(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right] \\ &\quad = t^{d_D} \sum_{m\geq 0} \left[\sum_{m_1+m_2=m} \frac{\left(\partial_\epsilon^{m_1} \epsilon^{d_D}\right)(0)}{m_1!} \cdot (t\partial_t)^{\delta_D} R_D(\partial_z) \frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right] \epsilon^m, \\ &\epsilon^{\Delta_l} t^{d_l} a_l(z,\epsilon)(t\partial_t)^{\delta_l} R_l(\partial_z)\widehat{\mathbb{G}}_2(\epsilon) \\ &\quad = t^{d_l} \sum_{m\geq 0} \left[\sum_{m_1+m_2+m_3=m} \frac{\left(\partial_\epsilon^{m_1} \epsilon^{\Delta_l}\right)(0)}{m_1!} \times \left[\frac{(\partial_\epsilon^{m_2} a_l)(z,0)}{m_2!}\right] \times (t\partial_t)^{\delta_l} R_l(\partial_z) \frac{\mathbb{G}_{m_3,2}(t,z)}{m_3!}\right] \epsilon^m, \end{aligned} \tag{265}$$

along with

$$\begin{aligned} &c_1(z,\epsilon)\widehat{\mathbb{G}}_2(\epsilon) = \sum_{m\geq 0} \left[\sum_{m_1+m_2=m} \left[\frac{(\partial_\epsilon^{m_1} c_1)(z,0)}{m_1!}\right] \times \frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right] \epsilon^m, \\ &\left[Q_1(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right] \times \left[Q_2(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right] \\ &\quad = \sum_{m\geq 0} \left[\sum_{m_1+m_2=m} \left[\frac{Q_1(\partial_z)\mathbb{G}_{m_1,2}(t,z)}{m_1!}\right] \times \left[\frac{Q_2(\partial_z)\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right]\right] \epsilon^m. \end{aligned} \tag{266}$$

As a result, relation (262) and the above formal expansions prompt the next partial differential equation satisfied by $\widehat{\mathbb{G}}_2(\epsilon)$,

$$
\begin{aligned}
Q(\partial_z)\widehat{\mathbb{G}}_2(\epsilon) = {} & (\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right] \\
& + \sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)(t\partial_t)^{\delta_l}R_l(\partial_z)\widehat{\mathbb{G}}_2(\epsilon) \\
& + F_2(\epsilon t,z,\epsilon) + c_1(z,\epsilon)\widehat{\mathbb{G}}_2(\epsilon) \\
& + c_{Q_1,Q_2}\left[Q_1(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]\times\left[Q_2(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right].
\end{aligned} \tag{267}
$$

In the next part of the proof, we exhibit recursion relations for the coefficients $G_{m,1}(t,z)$, $m \geq 0$. We proceed by taking the $m-$th derivative of both handsides of (257) with respect to $\epsilon$ for any given integer $m \geq 0$. Indeed, the Leibniz rule yields

$$
\begin{aligned}
& Q(\partial_z)\partial_\epsilon^m u_{1,p}(t,z,\epsilon) \\
& = \sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[\partial_\epsilon^{m_1}\epsilon^{d_D}\right]t^{d_D} \\
& \times\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\left[\partial_\epsilon^{m_2}u_{1,p}(t,z,\epsilon)\right]\right. \\
& \left.+\delta_D(t\partial_t)^{\delta_D-1}R_D(\partial_z)\left[\partial_\epsilon^{m_2}u_{2,p}(t,z,\epsilon)\right]\right] \\
& + \sum_{l=1}^{D-1}\sum_{m_1+m_2+m_3=m}\frac{m!}{m_1!m_2!m_3!}\times\left[\partial_\epsilon^{m_1}\epsilon^{\Delta_l}\right]t^{d_l} \\
& \times\left[(\partial_\epsilon^{m_2}a_l)(z,\epsilon)\right]\times\left[(t\partial_t)^{\delta_l}R_l(\partial_z)\left[\left(\partial_\epsilon^{m_3}u_{1,p}\right)(t,z,\epsilon)\right]\right. \\
& \left.+\delta_l(t\partial_t)^{\delta_l-1}R_l(\partial_z)\left[\left(\partial_\varepsilon^{m_3}u_{2,p}\right)(t,z,\varepsilon)\right]\right] + \partial_\varepsilon^m(F_1(\varepsilon t,z,\varepsilon)) \\
& + \sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[(\partial_\epsilon^{m_1}b_1)(z,\epsilon)\right]\times\left[\left(\partial_\epsilon^{m_2}u_{1,p}\right)(t,z,\epsilon)\right] \\
& + \sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[(\partial_\epsilon^{m_1}b_2)(z,\epsilon)\right]\times\left[\left(\partial_\epsilon^{m_2}u_{2,p}\right)(t,z,\epsilon)\right] \\
& + c_{P_1P_2}\sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[P_1(\partial_z)\left(\partial_\epsilon^{m_1}u_{1,p}\right)(t,z,\epsilon)\right] \\
& \times\left[P_2(\partial_z)\left(\partial_\epsilon^{m_2}u_{2,p}(t,z,\epsilon)\right]\right. \\
& + c_{P_3P_4}\sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[P_3(\partial_z)\left(\partial_\epsilon^{m_1}u_{1,p}\right)(t,z,\epsilon)\right] \\
& \times\left[P_4(\partial_z)\left(\partial_\epsilon^{m_2}u_{1,p}(t,z,\epsilon)\right]\right. \\
& + c_{P_5P_6}\sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[P_5(\partial_z)\left(\partial_\epsilon^{m_1}u_{2,p}\right)(t,z,\epsilon)\right] \\
& \times\left[P_6(\partial_z)\left(\partial_\epsilon^{m_2}u_{2,p}\right)(t,z,\epsilon)\right],
\end{aligned} \tag{268}
$$

for all $m \geq 0$, all $t \in \mathcal{T}$, $z \in H_{\beta'}$ and $\epsilon \in \mathcal{E}_p$. Besides, the asymptotic expansion (246) for $j = 1$ warrants the application of Proposition 25 in order to reach the limits

$$
\lim_{\substack{\epsilon\longrightarrow 0\\ \epsilon\in\mathcal{E}_p}}\ \sup_{\substack{t\in\mathcal{T}\\ z\in H_{\beta'}}}\left|\partial_\epsilon^m u_{1,p}(t,z,\epsilon) - \mathbb{G}_{m,1}(t,z)\right| = 0, \tag{269}
$$

for all integers $m \geq 0$ and any prescribed $0 \leq p \leq \varsigma - 1$. We allow the parameter $\varepsilon$ to get close to 0 in relation (268). Based on the above limits, (269) combined with (261) and the fact that the maps $u_{j,p}(t,z,\epsilon)$ and $\mathbb{G}_{m,j}(t,z)$, $j = 1, 2$ rely holomorphically in the variable $(t,z)$ on the product $\mathcal{T} \times H_{\beta'}$, we obtain the next relation for the coefficients $\mathbb{G}_{m,1}(t, z)$, $m \geq 0$,

$$
\begin{aligned}
& Q(\partial_z)\mathbb{G}_{m,1}(t,z) \\
& = \sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[\left(\partial_\epsilon^{m_1}\epsilon^{d_D}\right)(0)\right]t^{d_D} \\
& \times\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\mathbb{G}_{m_2,1}(t,z)\right. \\
& \left.+\delta_D(t\partial_t)^{\delta_D-1}R_D(\partial_z)\mathbb{G}_{m_2,2}(t,z)\right] \\
& + \sum_{l=1}^{D-1}\sum_{m_1+m_2+m_3=m}\frac{m!}{m_1!m_2!m_3!}\times\left[\left(\partial_\epsilon^{m_1}\epsilon^{\Delta_l}\right)(0)\right]t^{d_l} \\
& \times\left[(\partial_\epsilon^{m_2}a_l)(z,0)\right]\times\left[(t\partial_t)^{\delta_l}R_l(\partial_z)\mathbb{G}_{m_3,1}(t,z)\right. \\
& \left.+\delta_l(t\partial_t)^{\delta_l-1}R_l(\partial_z)\mathbb{G}_{m_3,2}(t,z)\right] + \partial_\epsilon^m(F_1(\epsilon t,z,\epsilon))_{|\epsilon=0} \\
& + \sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[(\partial_\epsilon^{m_1}b_1)(z,0)\right]\times\mathbb{G}_{m_2,1}(t,z) \\
& + \sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[(\partial_\epsilon^{m_1}b_2)(z,0)\right]\times\mathbb{G}_{m_2,2}(t,z) \\
& + c_{P_1P_2}\sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[P_1(\partial_z)\mathbb{G}_{m_1,1}(t,z)\right] \\
& \times\left[P_2(\partial_z)\mathbb{G}_{m_2,2}(t,z)\right] \\
& + c_{P_3P_4}\sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[P_3(\partial_z)\mathbb{G}_{m_1,1}(t,z)\right] \\
& \times\left[P_4(\partial_z)\mathbb{G}_{m_2,1}(t,z)\right] \\
& + c_{P_5P_6}\sum_{m=m_1+m_2}\frac{m!}{m_1!m_2!}\left[P_5(\partial_z)\mathbb{G}_{m_1,2}(t,z)\right] \\
& \times\left[P_6(\partial_z)\mathbb{G}_{m_2,2}(t,z)\right],
\end{aligned} \tag{270}
$$

for all $m \geq 0$, whenever $t \in \mathcal{T}$ and $z \in H_{\beta'}$.

This latter recursion relation leads to some partial differential equation governing the formal expression $\widehat{\mathbb{G}}_1(\epsilon)$ given by (245). In the process, we use the convergent Taylor expansions (263) together with

$$
\begin{aligned}
F_1(\epsilon t,z,\epsilon) &= \sum_{m\geq 0}\frac{\partial_\epsilon^m F_1(\epsilon t,z,\epsilon)_{|\epsilon=0}}{m!}\epsilon^m, \\
b_j(z,\epsilon) &= \sum_{m\geq 0}\frac{\left(\partial_\epsilon^m b_j\right)(z,0)}{m!}\epsilon^m,
\end{aligned} \tag{271}
$$

for $j=1,2$ which are valid for all $\epsilon \in D_{\epsilon_0}$ and from which the next list of computations are deduced

$$(\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)+\delta_D(t\partial_t)^{\delta_D-1}R_D(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]$$
$$=t^{d_D}\sum_{m\geq 0}\left[\sum_{m=m_1+m_2}\frac{(\partial_\epsilon^{m_1}\epsilon^{d_D})(0)}{m_1!}\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\frac{\mathbb{G}_{m_2,1}(t,z)}{m_2!}\right.\right.$$
$$\left.\left.+\delta_D(t\partial_t)^{\delta_D-1}R_D(\partial_z)\frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right]\right]\epsilon^m,$$
$$\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)\left[(t\partial_t)^{\delta_l}R_l(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)+\delta_l(t\partial_t)^{\delta_l-1}R_l(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]$$
$$=t^{d_l}\sum_{m\geq 0}\left[\sum_{m_1+m_2+m_3=m}\frac{(\partial_\epsilon^{m_1}\epsilon^{\Delta_l})(0)}{m_1!}\times\frac{(\partial_\epsilon^{m_2}a_l)(z,0)}{m_2!}\right.$$
$$\times\left[(t\partial_t)^{\delta_l}R_l(\partial_z)\frac{\mathbb{G}_{m_3,1}(t,z)}{m_3!}\right.$$
$$\left.\left.+\delta_l(t\partial_t)^{\delta_l-1}R_l(\partial_z)\frac{\mathbb{G}_{m_3,2}(t,z)}{m_3!}\right]\right]\epsilon^m, \tag{272}$$

along with

$$b_j(z,\epsilon)\widehat{\mathbb{G}}_j(\epsilon)=\sum_{m\geq 0}\left[\sum_{m=m_1+m_2}\frac{(\partial_\epsilon^{m_1}b_j)(z,0)}{m_1!}\times\frac{\mathbb{G}_{m_2,j}(t,z)}{m_2!}\right]\epsilon^m, \tag{273}$$

for $j=1,2$. Futhermore, the next identities hold

$$\left[P_1(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right]\times\left[P_2(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]$$
$$=\sum_{m\geq 0}\left[\sum_{m=m_1+m_2}\left[P_1(\partial_z)\frac{\mathbb{G}_{m_1,1}(t,z)}{m_1!}\right]\right.$$
$$\left.\times\left[P_2(\partial_z)\frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right]\right]\epsilon^m, \tag{274}$$

with

$$\left[P_3(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right]\times\left[P_4(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right]$$
$$=\sum_{m\geq 0}\left[\sum_{m=m_1+m_2}\left[P_3(\partial_z)\frac{\mathbb{G}_{m_1,1}(t,z)}{m_1!}\right]\right.$$
$$\left.\times\left[P_4(\partial_z)\frac{\mathbb{G}_{m_2,1}(t,z)}{m_2!}\right]\right]\epsilon^m,$$
$$\left[P_5(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]\times\left[P_6(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]$$
$$=\sum_{m\geq 0}\left[\sum_{m=m_1+m_2}\left[P_5(\partial_z)\frac{\mathbb{G}_{m_1,2}(t,z)}{m_1!}\right]\right.$$
$$\left.\times\left[P_6(\partial_z)\frac{\mathbb{G}_{m_2,2}(t,z)}{m_2!}\right]\right]\epsilon^m, \tag{275}$$

As a consequence of the above computations, relation (270) triggers the next partial differential equation fulfilled by $\widehat{\mathbb{G}}_1(\epsilon)$ and coupled with (267),

$$Q(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)$$
$$=(\epsilon t)^{d_D}\left[(t\partial_t)^{\delta_D}R_D(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)+\delta_D(t\partial_t)^{\delta_D-1}R_D(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]$$
$$+\sum_{l=1}^{D-1}\epsilon^{\Delta_l}t^{d_l}a_l(z,\epsilon)\left[(t\partial_t)^{\delta_l}R_l(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right.$$
$$\left.+\delta_l(t\partial_t)^{\delta_l-1}R_l(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]+F_1(\epsilon t,z,\epsilon)$$
$$+b_1(z,\epsilon)\widehat{\mathbb{G}}_1(\epsilon)+b_2(z,\epsilon)\widehat{\mathbb{G}}_2(\epsilon)$$
$$+c_{P_1P_2}\left[P_1(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right]\times\left[P_2(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]$$
$$+c_{P_3P_4}\left[P_3(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right]\times\left[P_4(\partial_z)\widehat{\mathbb{G}}_1(\epsilon)\right]$$
$$+c_{P_5P_6}\left[P_5(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]\times\left[P_6(\partial_z)\widehat{\mathbb{G}}_2(\epsilon)\right]. \tag{276}$$

In conclusion, we have checked by means of (267) that the power series $\widehat{\mathbb{G}}_2(\epsilon)$ formally solves the same partial differential equations as the function $u_{2,p}(t,z,\epsilon)$ stated in (256). In addition, through (276) and (257), we observe that the formal power series $\widehat{\mathbb{G}}_1(\epsilon)$ and the map $u_{1,p}(t,z,\epsilon)$ obey identical coupled partial differential equations. Then, drew on the computations (65), (63) and (62) performed reversedly from Subsection 3.1, we deduce that the formal expression $\widehat{\mathbb{G}}(\epsilon)$ stated in (252) conforms the same equation as the analytic map $u_p(t,z,\epsilon)$ given in (248) and recast as (253) where the formal monodromy operator around 0 given by $\gamma_\epsilon^*$ acts on the formal expression $\widehat{\mathbb{G}}(\epsilon)$ by dint of the formula (32) in Definition 1. This completes the proof of the third item of Theorem 24.

## 9. Conclusion and Perspectives

In this work, we have considered an initial value problem that is singularly perturbed in a complex parameter and possesses an irregular singularity in a complex time at the origin. This problem involves specific nonlocal nonlinearities, where the so-called formal monodromy operator plays a central role and can be viewed as a mixed-type partial differential and difference nonlinear equation when aiming at potential applications.

The presence of these nonlocal monodromy operators, which is the main novelty of our approach, enables the construction of holomorphic solutions with logarithmic expansions of finite type given by (12) or (20) in a more general setting. It is worth noting that in the case of polynomial nonlinearities involving only powers of the solution and its derivatives, logarithmic expansions of infinite type might appear (i.e., infinitely many powers of $\log(\epsilon t)$ might be encountered), as shown in the studies [18, 24] quoted in the introduction of this work.

The terms comprising the nonlinear part of our problem are suitably selected in a way that the construction of logarithmic-type solutions can be reduced to the study of analytic solutions to a coupling of two singularly perturbed initial value problems with quadratic local nonlinearities with a particularly simple shape of triangular form. The treatment of more general nonlinearities leading to nontriangular structures looks far more challenging and is postponed to a future investigation.

At last, we expect that our approach can be adapted to other related problems, for instance in the context of $q-$ difference equations which is another field of research of the author and his colleagues.

## Data Availability

There a no underlying data supporting the results of the study.

## Disclosure

The present work is registered as a preprint on http://preprints.org where it is quoted as [38].

## Conflicts of Interest

The author declares that they have no conflicts of interest.

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