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Research Article

Stability Results for Enriched Contraction Mappings in Convex Metric Spaces

Rekha Panicker and **Rahul Shukla**

Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa

Correspondence should be addressed to Rekha Panicker; rmpanicker71@gmail.com

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In this paper, we obtain some stability results of fixed point sets for a sequence of enriched contraction mappings in the setting of convex metric spaces. In particular, two types of convergence of mappings, namely, $(\mathscr{G})$-convergence and (H)-convergence are considered. We also illustrate our results by an application to an initial value problem for an ordinary differential equation.

1. Introduction

Stability is a term that refers to the limiting characteristics of a system. The stability of fixed points is the study of the relationship between the convergence of a sequence of mappings on a metric space or a Banach space and the sequence of its fixed points. If the fixed point sets of a sequence of mappings converge to the set of fixed points of their limit mappings, the sequence is said to be stable. We recommend the references [1–8] for some interesting results on the stability of fixed point sets of families of mappings in various settings. In this connection, see also the paper by Reich and Shafrir [9]. Pointwise and uniform convergence play a significant role in the study of fixed point stability. The preceding concepts, however, do not hold true when the domain of definition of all mappings in the study is neither the same space nor a unique nonempty subset of it. Barbet and Nachi [10] overcame this problem by introducing two new ideas of convergence called $(\mathscr{G})$-convergence and (H)-convergence, which they used to derive stability results in metric spaces. These results generalize the corresponding results of Bonsall [2], Fraser and Nadler [4], and Nadler [6]. The Barbet-Nachi work is unique in the sense that it redefines pointwise and uniform convergence for operators defined on the subsets of a space rather than the entire metric space. $(\mathscr{G})$-convergence generalized pointwise convergence, while (H)-convergence extended uniform convergence. Afterwards, many authors generalized these results in different settings for various type of mappings (see, for example, [11–15]).

On the other hand, Takahashi [16] considered the concept of convex metric space and looked at a few fixed point theorems for nonexpansive mappings in this space. He observed that the concept of fixed point theorems in Banach spaces can be extended to convex metric spaces. Subsequently, Ciric [17], Naimpally et al. [18], and many others studied fixed point theorems for different classes of mappings in convex metric spaces as well as a subclass of it (compare [19, 20]). In addition, Berinde and Păcurar [21, 22] obtained some fixed point theorems for a class of enriched contraction mappings.

In this paper, we study some convergence results in the sense of Barbet and Nachi [10] for a sequence of enriched contraction mappings and their fixed points in convex metric spaces as well as a subclass of it.

2. Preliminaries

We present some definitions, notions, and facts from the literature in this section. Throughout this paper, we use the notation $\mathbb{N}$ to denote the set of real numbers and $\bar{\mathbb{N}}$ to denote $\mathbb{N} \cup \{\infty\}$.

Definition 1 (see [16]). Let $(\mathscr{B}, d)$ be a metric space. A continuous function $W : \mathscr{B}^2 \times [0, 1] \longrightarrow \mathscr{B}$ is said to be a convex structure on $\mathscr{B}$ if, for all $\vartheta, \nu \in \mathscr{B}$ and any $\lambda \in [0,1]$,

$$d(u, W(\vartheta, \nu, \lambda)) \leq \lambda d(u, \vartheta) + (1-\lambda) d(u, \nu), \tag{1}$$

for any $u \in \mathscr{B}$.

A metric space $(\mathscr{B}, d)$ endowed with a convex structure W is called a convex metric space and is usually denoted by $(\mathscr{B}, d, W)$.

Obviously, any linear normed space and each of its convex subsets are convex metric spaces, with the natural convex structure

$$W(\vartheta, \nu, \lambda) = \lambda \vartheta + (1-\lambda) \nu, \tag{2}$$

for all $\vartheta, \nu \in \mathscr{B}$ and $\lambda \in [0,1]$. However, the converse is not true: there are a number of examples of convex metric spaces that cannot be nested in any Banach space [16]. Let $(\mathscr{B}, d, W)$ be a convex metric space and $\xi : \mathscr{B} \longrightarrow \mathscr{B}$ be a self mapping. $\mathrm{Fix}(\xi)$ denotes the set of all fixed points of ξ, that is,

$$\mathrm{Fix}(\xi) = \{\vartheta \in \mathscr{B} | \xi(\vartheta) = \vartheta\}. \tag{3}$$

Let $(\mathscr{B}, d)$ be a metric space and $[0, l] \subset \mathbb{R}$. A mapping $c : [0, l] \longrightarrow \mathscr{B}$ is called as geodesic path from ϑ to ν if

$$c(0) = \vartheta, c(l) = \nu, d\left(c(t), c\left(t'\right)\right) = \left|t - t'\right|, \tag{4}$$

for every $t, t' \in [0, l]$. The image $c([0, l])$ of c forms a geodesic joining ϑ and ν. It is noted that the geodesic segment joining ϑ and ν is not necessarily unique. In some results of this paper, we shall deal with a subclass of convex metric spaces called W-hyperbolic metric space (or W-hyperbolic space) in the sense of Kohlenbach [23].

Definition 2 (see [23]). The triplet $(\mathscr{B}, d, W)$ is called a hyperbolic metric space if $(\mathscr{B}, d)$ is a metric space and the function $W : \mathscr{B} \times \mathscr{B} \times [0,1] \longrightarrow \mathscr{B}$ satisfies the following conditions for all $\vartheta, \nu, z, w \in \mathscr{B}$ and $\mu, \eta \in [0,1]$:

(W1) $d(z, W(\vartheta, \nu, \mu)) \leq (1-\mu) d(z, \vartheta) + \mu d(z, \nu)$,
(W2) $d(W(\vartheta, \nu, \mu), W(\vartheta, \nu, \eta)) = |\mu - \eta| d(\vartheta, \nu)$,
(W3) $W(\vartheta, \nu, \mu) = W(\nu, \vartheta, 1-\mu)$,
(W4) $d(W(\vartheta, z, \mu), W(\nu, w, \mu)) \leq (1-\mu) d(\vartheta, \nu) + \mu d(z, w)$.

Remark 3. If $W(\vartheta, \nu, \mu) = (1-\mu)\vartheta + \mu \nu$ for all $\vartheta, \nu \in \mathscr{B}, \mu \in [0,1]$, then it can be seen that all normed linear spaces are included in these spaces.

Remark 4. If conditions (W1)–(W3) are satisfied, then $(\mathscr{B}, d, W)$ is the hyperbolic type space considered by Goebel and Kirk [24].

We shall write

$$W(\vartheta, \nu, \mu) \coloneqq (1-\mu)\vartheta \oplus \mu \nu, \tag{5}$$

to indicate the point $W(\vartheta, \nu, \mu)$ in a given hyperbolic metric space.

3. (b, θ)-Enriched Contraction Mapping

First, we recall the following definitions and results from Berinde and Păcurar [21] which play an important role in this paper.

Definition 5. Let $(\mathscr{B}, \|.\|)$ be a Banach space. A mapping $\xi : \mathscr{B} \longrightarrow \mathscr{B}$ is said to be a (b, θ) -enriched contraction mapping if there exist $b \in [0, \infty)$ and $\theta \in [0, b+1)$ such that for all $\vartheta, \nu \in \mathscr{B}$,

$$\|b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)\| \leq \theta \|\vartheta - \nu\|. \tag{6}$$

Remark 6. It is pointed out in [21] that every contraction mapping ξ is a $(0, \theta)$ -enriched mapping. The class of quasicontraction mapping and the class of (b, θ) -enriched contraction mappings are independent in nature. We present some examples to support our claim.

Example 1 (see [21]). Let $\mathscr{C} = [0,1] \subset \mathbb{R}$ with the usual metric and $\xi : \mathscr{C} \longrightarrow \mathscr{C}$ be a mapping defined as
$\xi(\vartheta) = 1 - \vartheta$ for all $\vartheta \in \mathscr{C}$.

Then, $F(\xi) = \{1/2\}$. ξ is a $(b, 1-b)$-enriched contraction mapping for any $b \in (0,1)$. At $\vartheta = 0.4$ and $\nu = 0.6$,

$$\begin{aligned} d(\xi(\vartheta), \xi(\nu)) = 0.2 > k & \max \{0.2, 0.2, 0.2, 0, 0\} \\ &= k \max \{|0.4 - 0.6|, |0.4 - 0.6|, |0.6 - 0.4|, |0.4 - 0.4|, |0.6, 0.6|\} \\ &= k \max \{d(\vartheta, \nu), d(\vartheta, \xi(\vartheta)), d(\nu, \xi(\nu)), d(\vartheta, \xi(\nu)), d(\nu, \xi(\vartheta))\}, \end{aligned} \tag{7}$$

for any $k \in [0,1)$. Thus, ξ is not a quasicontraction mapping.

Example 2. Let $\mathscr{K} = [0,1] \subset \mathbb{R}$ with the usual metric and $\xi : \mathscr{K} \longrightarrow \mathscr{K}$ be a mapping defined as

$$\xi(\vartheta) = \begin{cases} \dfrac{1}{4}, & \text{if } \vartheta \neq 1, \\ \dfrac{1}{2}, & \text{if } \vartheta = 1. \end{cases} \tag{8}$$

Then, $F(\xi) = \{1/4\}$. Now, we show that ξ is a quasicontraction mapping. If $\vartheta \neq 1$ and $\nu = 1$, then

$$\begin{aligned} d(\xi(\vartheta), \xi(\nu)) &= \left|\frac{1}{4} - \frac{1}{2}\right| = \frac{1}{4} \leq k \frac{3}{4} \\ &\leq k \max \left\{|1 - \vartheta|, \left|\frac{1}{4} - \vartheta\right|, \left|1 - \frac{1}{2}\right|, \left|\frac{1}{2} - \vartheta\right|, \left|1 - \frac{1}{4}\right|\right\} \\ &= k \max \{d(\vartheta, \nu), d(\vartheta, \xi(\vartheta)), d(\nu, \xi(\nu)), d(\vartheta, \xi(\nu)), d(\nu, \xi(\vartheta))\}, \end{aligned} \tag{9}$$

for any $k \geq 1/3$. On the other hand, at $\vartheta = 0.9$ and $\nu = 1$,

$$\begin{aligned} |b(\vartheta - \nu) + \xi(\vartheta) - \xi(\nu)| &= \left| b(0.9-1) + \left(\frac{1}{4} - \frac{1}{2}\right) \right| \\ &= \left| b(-0.1) - \frac{1}{4} \right| > (b+1)(0.1) \\ &> \theta|\vartheta - \nu|. \end{aligned} \tag{10}$$

ξ is not a b-enriched contraction mapping for any $b \in [0,\infty)$.

Now, we consider Definition 5 in convex spaces:

Definition 7 (see [6]). Let $(\mathscr{B}, d, W)$ be a convex metric space. A mapping $\xi : \mathscr{B} \longrightarrow \mathscr{B}$ is said to be an enriched contraction mapping if there exist $c \in [0,1)$ and $\lambda \in [0,1)$ such that

$$d(W(\vartheta, \xi(\vartheta), \lambda), W(\nu, \xi(\nu), \lambda)) \leq c d(\vartheta, \nu) \quad \text{for all } \vartheta, \nu \in \mathscr{B}. \tag{11}$$

Define the mapping $S : \mathscr{B} \longrightarrow \mathscr{B}$ by $S(\vartheta) = W(\vartheta, \xi(\vartheta), \lambda) = (1-\lambda)\vartheta + \lambda\xi(\vartheta)$. Then, S is a Banach contraction, and hence, it has a fixed point in complete metric spaces.

4. Barbet-Nachi-Type Convergence

Following the conditions developed by Barbet and Nachi [10], we define the following conceptions of convergence for enriched contraction mappings in convex metric spaces, which generalize the notions of pointwise and uniform convergence.

Let $\{\mathscr{B}_n\}_{n\in\bar{\mathbb{N}}}$ be a family of nonempty subsets of the convex metric space $\mathscr{B}$ and $\{\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ a family of enriched contraction mappings [10]. Then, ξ_∞ is said to be a $(\mathscr{G})$-limit of the sequence $\{\xi_n\}_{n\in\mathbb{N}}$, or equivalently, $\{\xi_n\}_{n\in\bar{\mathbb{N}}}$ satisfies the property $(\mathscr{G})$ if the following condition holds:

$(\mathscr{G})$: $Gr(\xi_\infty) \subset \lim\inf Gr(\xi_n)$: for every $\vartheta \in \mathscr{B}_\infty$, there exists a sequence $\{\vartheta_n\}$ in $\prod_{n\in\mathbb{N}}\mathscr{B}_n$ such that

$$\lim_n d(\vartheta_n, \vartheta) = 0, \lim_n d(\xi_n\vartheta_n, \xi_\infty\vartheta) = 0, \tag{12}$$

where $Gr(\xi)$ stands for the graph of ξ.

Further, ξ_∞ is a (H)-limit of the sequence of enriched contraction mappings $\{\xi_n\}_{n\in\mathbb{N}}$, or equivalently, $\{\xi_n\}_{n\in\bar{\mathbb{N}}}$ satisfies the property (H) if the following condition holds [10]:

(H): For all sequences $\{\vartheta_n\}$ in $\prod_{n\in\mathbb{N}}\mathscr{B}_n$, there exists a sequence $\{\nu_n\}$ in $\mathscr{B}_\infty$ such that

$$\lim_n d(\vartheta_n, \nu_n) = 0, \lim_n d(\xi_n\vartheta_n, \xi_\infty\nu_n) = 0. \tag{13}$$

Lemma 8. *Let $(\mathscr{B}, d, W)$ be a convex metric space, $(\mathscr{B}_n)_{n\in\bar{\mathbb{N}}}$ a family of nonempty subsets of $\mathscr{B}$, and $\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}$ a sequence of enriched contraction mappings. Define the mapping $S_n : \mathscr{B}_n \longrightarrow \mathscr{B}$ by*

$$S_n(\vartheta_n) = W(\vartheta_n, \xi_n(\vartheta_n), \lambda), \tag{14}$$

where $(\vartheta_n) \in \prod_{n\in\mathbb{N}}\mathscr{B}_n$. Then, for any $\lambda \in (0,1)$, $Fix(\xi_n) = Fix(S_n)$.

Proof. Let $a_n \in \mathrm{Fix}(\xi_n)$. That is, $\xi_n(a_n) = a_n$.

$$\begin{aligned} d(a_n, S_n(a_n)) &= d(a_n, W(a_n, \xi_n(a_n), \lambda) \\ &\leq \lambda d(a_n, a_n) + (1-\lambda) d(a_n, \xi_n(a_n)) = 0. \end{aligned} \tag{15}$$

That is, $a_n \in \mathrm{Fix}(S_n)$.

Conversely, assume $a_n \in \mathrm{Fix}(S_n)$. This means that $d(a_n, S_n(a_n)) = 0$, and hence, we have

$$d(a_n, W(a_n, \xi_n(a_n), \lambda) = 0 \Rightarrow (1-\lambda) d(a_n, \xi_n(a_n)) = 0. \tag{16}$$

But $(1-\lambda) \neq 0$, which implies $d(a_n, \xi_n(a_n)) = 0$.

That is, $a_n \in Fix(\xi_n)$. Hence $Fix(\xi_n) = Fix(S_n)$. □

5. Stability Results for $(\mathscr{G})$-Convergence

The following stability result improves and generalizes the results of Fraser and Nadler [4] and Nachi [25].

Theorem 9. *Let $(\mathscr{B}, d, W)$ be a convex metric space and $\{\mathscr{B}_n\}_{n\in\bar{\mathbb{N}}}$ a family of nonempty subsets of $\mathscr{B}$. $\{\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ is a family of enriched contraction mappings satisfying the property $(\mathscr{G})$. If for all $n \in \bar{\mathbb{N}}$, ϑ_n is a fixed point of ξ_n, then the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ converges to ϑ_∞.*

Proof. Since ξ_n satisfies property $(\mathscr{G})$, $\forall \vartheta_\infty \in \mathscr{B}_\infty$, there exists a sequence $\{\nu_n\}$ in $\prod_{n\in\mathbb{N}}\mathscr{B}_n$ such that $\nu_n \longrightarrow \vartheta_\infty$ and $\xi_n(\nu_n) \longrightarrow \xi_\infty(\vartheta_\infty)$. Thus,

$$d(\vartheta_n, \vartheta_\infty) = d(\xi_n(\vartheta_n), \xi_\infty(\vartheta_\infty)). \tag{17}$$

But $\mathrm{Fix}(\xi_n) = Fix(S_n)$. Therefore,

$$\begin{aligned} d(\vartheta_n, \vartheta_\infty) &= d(S_n(\vartheta_n), \xi_\infty(\vartheta_\infty)) \\ &\leq d(S_n(\vartheta_n), S_n(\nu_n)) + d(S_n(\nu_n), \xi_n(\nu_n)) \\ &\quad + d(\xi_n(\nu_n), \xi_\infty(\vartheta_\infty)) \\ &\leq c d(\vartheta_n, \nu_n) + d(W(\nu_n, \xi_n(\nu_n), \lambda), \xi_n(\nu_n)) \\ &\quad + d(\xi_n(\nu_n), \xi_\infty(\vartheta_\infty)) \\ &\leq c[d(\vartheta_n, \vartheta_\infty) + d(\nu_n, \vartheta_\infty)] + \lambda d(\nu_n, \xi_n(\nu_n)) \\ &\quad + d(\xi_n(\nu_n), \xi_\infty(\vartheta_\infty)), \end{aligned}$$

$$\begin{aligned} (1-c) d(\vartheta_n, \vartheta_\infty) &\leq c d(\nu_n, \vartheta_\infty) \\ &\quad + \lambda[d(\nu_n, \xi_\infty(\vartheta_\infty)) + d(\xi_n(\nu_n), \xi_\infty(\vartheta_\infty))] \\ &\quad + d(\xi_n(\nu_n), \xi_\infty(\vartheta_\infty)). \end{aligned} \tag{18}$$

Taking the limit as $n \longrightarrow \infty$, we obtain

$$\lim_{n\longrightarrow\infty}(1-c)d(\vartheta_n, \vartheta_\infty) \le 0. \tag{19}$$

Since $0 \le c < 1$, we get $\lim_{n\longrightarrow\infty} d(\vartheta_n, \vartheta_\infty) = 0$. □

Corollary 10. *Let $(\mathscr{B}, d, W)$ be a convex metric space and $\{\xi_n : \mathscr{B} \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ a family of enriched contraction mappings satisfying the property $(\mathscr{G})$. If for all $n \in \bar{\mathbb{N}}$, ϑ_n is a fixed point of ξ_n, then the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ converges to ϑ_∞.*

Proof. By taking $\mathscr{B}_n = \mathscr{B}$ for $n \in \bar{\mathbb{N}}$ in Theorem 9, we find the desired conclusion easily. □

The following theorem proves the existence of a fixed point for a $(\mathscr{G})$-limit of a sequence of enriched contraction mappings.

Theorem 11. *Let $(\mathscr{B}, d, W)$ be a convex metric space and $\{\mathscr{B}_n\}_{n\in\bar{\mathbb{N}}}$ a family of nonempty subsets of $\mathscr{B}$. Suppose that $\{\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ is a sequence of enriched contraction mappings satisfying the property $(\mathscr{G})$. If $\vartheta_n \in \mathscr{B}_n$ is a fixed point of ξ_n for each $n \in \mathbb{N}$ and the sequence $\{\vartheta_n\}$ admits a subsequence converging to $\vartheta_\infty \in \mathscr{B}_\infty$, then ϑ_∞ is a fixed point of ξ_∞.*

Proof. Let $\{\vartheta_{n_k}\}$ be a subsequence of $\{\vartheta_n\}$ such that $\lim_{k\longrightarrow\infty} \vartheta_{n_k} = \vartheta_\infty \in \mathscr{B}_\infty$. By the property $(\mathscr{G})$, there exists a sequence $\{\nu_n\}$ in $\prod_{n\in\mathbb{N}}\mathscr{B}_n$ such that $d(\nu_n, \vartheta_\infty) = 0$ and $d(\xi_n(\nu_n), \xi_\infty(\vartheta_\infty)) = 0$. For any $k \in \mathbb{N}$, we have

$$\begin{aligned} d(\vartheta_\infty, \xi_\infty(\vartheta_\infty)) \le{}& d(\vartheta_\infty, \vartheta_{n_k}) + d(\xi_{n_k}(\vartheta_{n_k}), \xi_{n_k}(\nu_{n_k})) \\ &+ d(\xi_{n_k}(\nu_{n_k}), \xi_\infty(\vartheta_\infty)). \end{aligned} \tag{20}$$

Since ξ_{n_k} is an enriched contraction mapping,

$$d(S_{n_k}(\vartheta_{n_k}), S_{n_k}(\nu_{n_k})) \le c d(\vartheta_{n_k}, \nu_{n_k}). \tag{21}$$

But $\xi_{n_k}(\vartheta_{n_k}) = \vartheta_{n_k} = S_{n_k}(\vartheta_{n_k})$. Then, we have

$$\begin{aligned} d(\xi_{n_k}(\vartheta_{n_k}), \xi_{n_k}(\nu_{n_k})) &= d(S_{n_k}(\vartheta_{n_k}), \xi_{n_k}(\nu_{n_k})) \\ &\le d(S_{n_k}(\vartheta_{n_k}), S_{n_k}(\nu_{n_k})) + d(S_{n_k}(\nu_{n_k}), \xi_{n_k}(\nu_{n_k})) \\ &\le c d(\vartheta_{n_k}, \nu_{n_k}) + d(W(\nu_{n_k}, \xi_{n_k}(\nu_{n_k}), \lambda), \xi_{n_k}(\nu_{n_k})) \\ &= c d(\vartheta_{n_k}, \nu_{n_k}) + \lambda d(\nu_{n_k}, \xi_{n_k}(\nu_{n_k})) \\ &\le c[d(\vartheta_{n_k}, \vartheta_\infty) + d(\nu_{n_k}, \vartheta_\infty)] \\ &\quad + \lambda[d(\nu_{n_k}, \xi_\infty(\vartheta_\infty)) + d(\xi_{n_k}(\nu_{n_k}), \xi_\infty(\vartheta_\infty))]. \end{aligned} \tag{22}$$

Then, from (20), we have

$$\begin{aligned} d(\vartheta_\infty, \xi_\infty(\vartheta_\infty)) \le{}& d(\vartheta_\infty, \vartheta_{n_k}) + c[d(\vartheta_{n_k}, \vartheta_\infty) + d(\nu_{n_k}, \vartheta_\infty)] \\ &+ \lambda[d(\nu_{n_k}, \xi_\infty(\vartheta_\infty)) + d(\xi_{n_k}(\nu_{n_k}), \xi_\infty(\vartheta_\infty)) \\ &+ d(\xi_{n_k}(\nu_{n_k}), \xi_\infty(\vartheta_\infty))]. \end{aligned} \tag{23}$$

Letting $k \longrightarrow \infty$, we deduce that

$$\begin{aligned} d(\vartheta_\infty, \xi_\infty(\vartheta_\infty)) &\le \lambda \lim_{k\longrightarrow\infty} d(\nu_{n_k}, \xi_\infty(\vartheta_\infty)) \\ &\le \lambda\left[\lim_{k\longrightarrow\infty} d(\nu_{n_k}, \vartheta_\infty) + d(\vartheta_\infty, \xi_\infty(\vartheta_\infty))\right], \end{aligned} \tag{24}$$

which implies

$$d(\vartheta_\infty, \xi_\infty(\vartheta_\infty)) \le \lambda d(\vartheta_\infty, \xi_\infty(\vartheta_\infty)). \tag{25}$$

That is,

$$(1-\lambda)d(\vartheta_\infty, \xi_\infty(\vartheta_\infty)) \le 0 \tag{26}$$

.

But $\lambda \in (0, 1)$. Hence $\xi_\infty(\vartheta_\infty) = \vartheta_\infty$. □

Proposition 12. *Let $(\mathscr{B}, d, W)$ be a hyperbolic metric space, $\{\mathscr{B}_n\}_{n\in\bar{\mathbb{N}}}$ a family of nonempty subsets of $\mathscr{B}$, and $\{\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ a family of mappings satisfying the property $(\mathscr{G})$, such that, for any $n \in \mathbb{N}$, ξ_n is an enriched contraction mapping. Then, ξ_∞ is also an enriched contraction mapping.*

Proof. Let ϑ and ν be two elements in $\mathscr{B}_\infty$. By the property $(\mathscr{G})$, there exist two sequences $\{\vartheta_n\}$ and $\{\nu_n\}$ in $\prod_{n\in\mathbb{N}}\mathscr{B}_n$ converging, respectively, to ϑ and ν such that the sequences $\{\xi_n(\vartheta_n)\}$ and $\{\xi_n(\nu_n)\}$ converge to $\xi_\infty(\vartheta)$ and $\xi_\infty(\nu)$, respectively. For $\lambda = 1/(b+1)$, one can set

$$S_\infty(\vartheta) = W(\vartheta, \xi_\infty(\vartheta), \lambda), \tag{27}$$

for all $\vartheta \in \mathscr{B}_\infty$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(S_\infty(\vartheta), S_\infty(\nu)) &\le d(S_\infty(\vartheta), S_n(\vartheta_n)) + d(S_n(\vartheta_n), S_n(\nu_n)) \\ &\quad + d(S_n(\nu_n), S_\infty(\nu)) \\ &\le d(S_\infty(\vartheta), S_n(\vartheta_n)) + c d(\vartheta_n, \nu_n) \\ &\quad + d(S_n(\nu_n), S_\infty(\nu)). \end{aligned} \tag{28}$$

Now, we prove the following claim:

$$d(S_\infty(\vartheta), S_n(\vartheta_n)) \longrightarrow 0 \text{ and } d(S_n(\nu_n), S_\infty(\nu)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{29}$$

Given $\vartheta_n \longrightarrow \vartheta$ and $\xi_n(\vartheta_n) \longrightarrow \xi_\infty(\vartheta)$ as $n \longrightarrow \infty$. Then,

$$\begin{aligned} d(S_\infty\vartheta, S_n\vartheta_n) &= d(W(\vartheta, \xi_\infty(\vartheta), \lambda), W(\vartheta_n, \xi_n(\vartheta_n), \lambda)) \\ &\le (1-\lambda)d(\vartheta, \vartheta_n) + \lambda d(\xi_\infty(\vartheta), \xi_n(\vartheta_n)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{30}$$

Further, $\nu_n \longrightarrow \nu$ and $\xi_n(\nu_n) \longrightarrow \xi_\infty(\nu)$ as $n \longrightarrow \infty$. Thus,

$$\begin{aligned} d(S_n(\nu_n), S_\infty(\nu)) &= d(W(\nu_n, \xi_n(\nu_n), \lambda), W(\nu, \xi_\infty(\nu), \lambda)) \\ &\le (1-\lambda)d(\nu_n, \nu) \\ &\quad + \lambda d(\xi_n(\nu_n), \xi_\infty(\nu)) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{31}$$

Letting $n \longrightarrow \infty$ in (28), (30), and (31), we have

$$d(S_\infty(\vartheta), S_\infty(\nu)) \le c d(\vartheta, \nu), \tag{32}$$

and the conclusion holds. □

The following result in Proposition 4 of [10] follows from Proposition 12.

Corollary 13. *Let $\mathscr{B}$ be a metric space, $\{\mathscr{B}_n\}_{n\in\bar{\mathbb{N}}}$ a family of nonempty subsets of $\mathscr{B}$, and $\{\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ a family of mappings satisfying property $(\mathscr{G})$ such that, for any $n \in \mathbb{N}$, ξ_n is a k_n -contraction. Then, ξ_∞ is a k-contraction where $\{k_n\}$ is a bounded (resp.convergent) sequence with $=:\sup_n k_n$ (resp.$\lim_n k_n$).*

6. Stability Results for (H)-Convergence

Now, we present another stability result using the (H) -convergence.

Theorem 14. *Let $(\mathscr{B}, d\,; W)$ be a hyperbolic metric space and $\{\mathscr{B}_n\}_{n\in\bar{\mathbb{N}}}$ a family of nonempty subsets of $\mathscr{B}$. Let $\{\xi_n : \mathscr{B}_n \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ be a family of mappings satisfying the property (H) such that ξ_∞ is an enriched contraction mapping. If for each $n \in \bar{\mathbb{N}}$, $\vartheta_n \in \mathscr{B}_n$ is a fixed point of ξ_n, then the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ converges to ϑ_∞.*

Proof. By property (H), there exists a sequence $\{\nu_n\}$ in $\mathscr{B}_\infty$ such that $d(\vartheta_n, \nu_n) \longrightarrow 0$ and $d(\xi_n(\vartheta_n), \xi_\infty(\nu_n)) \longrightarrow 0$. ξ_∞ is an enriched contraction which implies

$$d(W(\vartheta, \xi_\infty(\vartheta), \lambda), W(\nu, \xi_\infty(\nu), \lambda)) \le c d(\vartheta, \nu) \quad \text{for all } \vartheta, \nu \in \mathscr{B}_\infty. \tag{33}$$

Given ϑ_n is a fixed point of ξ_n for each $n \in \bar{\mathbb{N}}$. Then, ϑ_n is a fixed point of S_n as well. That is, $S_n(\vartheta_n) = \vartheta_n$ for each $n \in \bar{\mathbb{N}}$.

Also,

$$\begin{aligned} d(S_n(\vartheta_n), S_\infty(\nu_n)) &= d((1-\lambda)\vartheta_n + \lambda\xi_n(\vartheta_n), (1-\lambda)\nu_n + \lambda\xi_\infty(\nu_n)) \\ &\le (1-\lambda)d(\vartheta_n, \nu_n) + \lambda(\xi_n(\vartheta_n), \xi_\infty(\nu_n)). \end{aligned} \tag{34}$$

By letting $n \longrightarrow \infty$, we see that $d(S_n(\vartheta_n), S_\infty(\nu_n)) = 0$. Now, by using the triangle inequality, we get

$$\begin{aligned} d(\vartheta_n, \vartheta_\infty) = d(S_n(\vartheta_n), S_\infty(\vartheta_\infty)) &\le d(S_n(\vartheta_n), S_\infty(\nu_n)) \\ &\quad + d(S_\infty(\nu_n), S_\infty(\vartheta_\infty)). \end{aligned} \tag{35}$$

As ξ_∞ is an enriched contraction mapping, S_∞ is a contraction mapping. Hence, by letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} \lim_{n\longrightarrow\infty} d(\vartheta_n, \vartheta_\infty) &\le \lim_{n\longrightarrow\infty} d(S_\infty(\nu_n), S_\infty(\vartheta_\infty)) \le \lim_{n\longrightarrow\infty} c d(\nu_n, \vartheta_\infty) \\ &\le \lim_{n\longrightarrow\infty} c[d(\vartheta_n, \nu_n) + d(\vartheta_n, \vartheta_\infty)] \\ &= c \lim_{n\longrightarrow\infty} d(\vartheta_n, \vartheta_\infty). \end{aligned} \tag{36}$$

Thus,

$$(1-c)\lim_{n\longrightarrow\infty} d(\vartheta_n, \vartheta_\infty) = 0, \tag{37}$$

and hence, the conclusion follows. □

Corollary 15. *Let $(\mathscr{B}, d, W)$ be a hyperbolic metric space and $\{\xi_n : \mathscr{B} \longrightarrow \mathscr{B}\}_{n\in\bar{\mathbb{N}}}$ be a family of mappings satisfying the property (H) converging to the enriched contraction mapping $\xi_\infty : \mathscr{B} \longrightarrow \mathscr{B}$. If for any $n \in \bar{\mathbb{N}}$, ϑ_n is a fixed point of ξ_n, then the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ converges to ϑ_∞.*

Proof. By taking $\mathscr{B}_n = \mathscr{B}$ for all $n \in \bar{\mathbb{N}}$ in Theorem 14, we obtain the desired result immediately. □

7. Application

We consider the following application of our results. The following theorem is motivated due to Nadler [6].

Theorem 16. *Let D be an open subset of $\mathbb{R}^2$, $(a, b) \in D$, $K \in \mathbb{R}$, and $K > 0$. Suppose that $\{\xi_i\}$ is a sequence of real valued continuous functions defined on D such that $|\xi_i(\vartheta, \nu)| \le K$ for all $(\vartheta, \nu) \in D$ with a $(\mathscr{G})$-limit ξ, a continuous function on D.*

The set $C = \{(\vartheta, \nu): |\vartheta - a| \le p$ and $|\nu - b| \le K|\vartheta - a|\}$ is a subset of D with $p > 0$.

Let $K_i \in \mathbb{R}$ and $K_i > 0$ for all $i \in \mathbb{N} \cup \{0\}$; $\{K_i\}$ be a bounded sequence. For all $(\vartheta, \nu),(\vartheta, z) \in D$, we have $|\xi_i(\vartheta, \nu) - \xi_i(\vartheta, z)| \le K_i|\nu - z|$ where $0 < K_i p < 1$ for all $i \in \mathbb{N} \cup \{0\}$.

Then, the sequence $\{\nu_i\}$ converges uniformly on $I = [a - p, a + p]$ to ν_0, where for each $i \in \mathbb{N} \cup \{0\}$,$\nu_i$ is the unique solution on I of the initial value problem

$$\nu'(\vartheta) = \xi_i(\vartheta, \nu(\vartheta))\,;\, \nu(a) = b. \tag{38}$$

Proof. Let $\mathscr{B}$ be the set of all real valued continuous functions on I with graph lying in C and with Lipschitz constant K. Then, $\mathscr{B}$ with the supremum norm $\|.\|$ is a compact Banach space. For each $g \in \mathscr{B}$, define

$$[A_i(g)](\vartheta) = b + \int_a^\vartheta \xi_i(t, g(t))dt \quad \forall \vartheta \in I. \tag{39}$$

For each $i \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} &|b(g(t)-h(t)) + [A_i(g)](\vartheta) - [A_i(h)](\vartheta)| \\ &= \left| b(g(t)-h(t)) + \int_a^\vartheta \xi_i(t, g(t))dt - \int_a^\vartheta \xi_i(t, h(t))dt \right| \\ &\le b|g(t)-h(t)| + K_i \int_a^\vartheta |g(t)-h(t)|dt \le b \sup_{t\in[a,\vartheta]} |g(t)-h(t)| \\ &+ K_i \sup_{t\in[a,\vartheta]} |g(t)-h(t)| \int_a^\vartheta dt \le b\|g-h\| + K_i p\|g-h\| = \theta_i \|g-h\|, \end{aligned} \tag{40}$$

where $\theta_i = b + K_i p \in [0, b+1)$ and A_i is a (b, θ_i) contraction mapping from $\mathscr{B}$ to $\mathscr{B}$ for all $i \in \mathbb{N} \cup \{0\}$. By Proposition 12, A is also an enriched contraction mapping on $\mathscr{B}$. For each $\vartheta \in I$, $g \in \mathscr{B}$, $i \in \mathbb{N} \cup \{0\}$,

$$[A_i(g)](\vartheta) - [A(g)](\vartheta) = \int_a^\vartheta [\xi_i(t, g(t)) - \xi_i(t, g(t))]dt. \tag{41}$$

Since ξ is the $(\mathscr{G})$-limit of ξ_i, the sequence of integrands converges to zero and is uniformly bounded by $2K$. The Lebesgue-bounded convergence theorem guarantees that the sequence of integrals on the R.H.S. goes to 0 as $i \longrightarrow \infty$. Therefore $A(g)$ is the $(\mathscr{G})$-limit of $A_i(g)$ on I. Now, by Proposition 4 of [26], the $(\mathscr{G})$-limit is equivalent to the pointwise limit. It is easy to see that $A_i(g)$ is uniformly continuous on I for each $i \in \mathbb{N} \cup \{0\}$, and hence, the sequence $\{A_i(g)\}$ is equicontinuous on the compact set I. Therefore, the sequence $\{A_i(g)\}$ converges uniformly to $A(g)$ on I. Hence, the sequence $\{A_i\}$ converges pointwise to A on $\mathscr{B}$. Since $\mathscr{B}$ is a compact Banach space, the sequence $\{\nu_i\}$ has a convergent subsequence $\{\nu_{i_j}\}$ converging to ν. By Theorem 11, ν is a fixed point of A. From Theorem 9, the sequence $\{\nu_i\}$, where ν_i is the unique fixed point of A_i for each $i \in \mathbb{N} \cup \{0\}$, converges to the fixed point ν of A. The result follows since these fixed points are the unique solutions of the initial value problem. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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