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Asymptotic estimates for Klein-Gordon equation on α-modulation space

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Abstract: Recently, asymptotic estimates for the unimodular Fourier multipliers $e^{i\mu(D)}$ have been studied for the function α-modulation space. In this paper, using the almost orthogonality of projections and some techniques on oscillating integrals, we obtain asymptotic estimates for the unimodular Fourier multiplier $e^{it(I-\Delta)^{\frac{\beta}{2}}}$ on the α-modulation space. For an application, we give the asymptotic estimate of the solution for the Klein-Gordon equation with initial data in a α-modulation space. We also obtain a quantitative form about the solution to the nonlinear Klein-Gordon equation.

Keywords: Unimodular multipliers, α-modulation spaces, Klein-gordon equation.

MSC: 42B15, 42B35, 42C15.

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1. Introduction

Suppose $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing smooth functions and tempered distributions, respectively, and the Fourier transform $\mathscr{F}(f) = \hat{f}$ and the inverse Fourier transform $\mathscr{F}^{-1}(f) = \check{f}$ of $f \in \mathscr{S}(\mathbb{R}^n)$ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx, \text{ and } \check{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}d\xi.$$

We define the Fourier multiplier is a linear operator H_μ defined on the set of test functions f on $\mathbb{R}^n$ is defined by

$$H_\mu f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mu(\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi.$$

The function μ is called the *symbol* or *multiplier* of H_μ. Note that the Fourier multiplier operator H_μ can be extended in the distribution sense with $\mu \in \mathscr{S}'(\mathbb{R}^n)$ by $H_\mu f = \mathscr{F}^{-1}(\mu\hat{f}) = (\mathscr{F}^{-1}\mu) * f$, for all $f \in \mathscr{S}(\mathbb{R}^n)$.

A fundamental question in the study of Fourier multipliers is to relate the boundedness properties of H_μ on certain function spaces to the properties of the symbol μ.

In this paper we will primarily focus on a particular Fourier multiplier, the *unimodular Fourier multipliers*, defined by the symbol of the type $\mu(\xi) = e^{i\lambda(\xi)}$, for real-valued functions λ. They arise when one solves the (half) Klein-Gordon equation

$$\begin{cases} i\partial_t u + (I-\Delta)^{\frac{\beta}{2}}u = 0, & \text{when } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where one has the formula solution $u(t,x) = \left(e^{it(I-\Delta)^{\frac{\beta}{2}}}u_0\right)(x)$. Here $\Delta = \Delta_x$ is the Laplacian and $e^{it(I-\Delta)^{\frac{\beta}{2}}}$ is the unimodular Fourier multiplier with the symbol $e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}$. The particular interest in studying this

Klein-Gordon type equation is by understanding the boundedness properties of the unimodular Fourier multiplier $e^{it(I-\Delta)^{\frac{\beta}{2}}}$ will provide insight to the behavior of the solution to the Klein-Gordon equation

$$\begin{cases} \partial_{tt}u + Iu - \Delta u = 0, & \text{when } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Further, to understand the behavior of the solution to the Klein-Gordon equation, we need to understand the behavior of the Fourier multiplier $\Theta_K(t)$ with symbol $\frac{\sin(t(1+|\xi|^2))^{\frac{1}{2}}}{(1+|\xi|^2)^{\frac{1}{2}}}$.

Unimodular Fourier multipliers generally do not preserve any Lebesgue space L^p, except for $p = 2$. Thus the L^p-spaces are not the appropriate function spaces for the study of unimodular Fourier multipliers. Thus we will focus on the function space α-modulation space, which is a generalization of the modulation space and Besov space.

[1] used the almost orthogonality of projections and some techniques on oscillating integrals to find bounded results for the unimodular Fourier multiplier $e^{it|\Delta|^{\frac{\beta}{2}}}$ on the modulation space. See [2] for additional information on resent developments on the modulation space. Recently, [3] used these methods to acquire similar bounded results for the unimodular Fourier multiplier $e^{it|\Delta|^{\frac{\beta}{2}}}$ on the α-modulation space showing that if $\beta = 1$, $t > 1$ and $1 \leq p, q, \leq \infty$, then

$$\left|\left|e^{it|\Delta|^{\frac{\beta}{2}}} f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+s_0,\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma \geq 0$ and s_0 is a constant depended on n and p, and if $\frac{1}{2} < \beta$ with $\beta \neq 1$, $t > 1$ and $1 \leq p, q \leq \infty$

$$\left|\left|e^{it|\Delta|^{\frac{\beta}{2}}} f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+s_1,\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma \geq 0$ and s_1 is a constant depended on β, α, n, and p.

In this paper, we use the same almost orthogonality of projections and techniques on oscillating integrals to find our results. Our main results can be stated as follows.

Theorem 1. *For $\beta \geq 1$, $1 \leq p, q \leq \infty$, and $t > 1$, then the following estimate holds:*

$$\left|\left|e^{it(I-\Delta)^{\frac{\beta}{2}}} f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M^{s+\nu_1(\beta,\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma > 0$ and $\nu_1(\beta, \alpha)$ is defined as

$$\nu_1(\beta, \alpha) = n(\beta - 2 + 2\alpha)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{1}$$

Theorem 2. *Let $1 \leq p, q \leq \infty$, and $t \geq 1$ then the following estimate holds:*

$$||\Theta_K(t)g||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M^{s+\nu_2(\alpha),\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma > 0$ is any positive number, and $\nu_2(\alpha)$ is defined by

$$\nu_2(\alpha) = (n\alpha - 2)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{2}$$

As an application of our theorems, we prove that the following Nonhomogeneous Klein-Gordon equation with initial date in an α-modulation space has a solution.

Theorem 3. *Consider the Nonlinear Klein-Gordon equation*

$$\begin{cases} \partial_{tt}u(t,x)+u(t,x)-\Delta u(t,x)+F(u(t,x))=0, & \text{for } (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ u(0,x)=f_u(x), & \text{for } x\in\mathbb{R}^n, \\ \partial_t u(0,x)=g_u(x), & \text{for } x\in\mathbb{R}^n, \end{cases}$$

where $F=|u|^{2k}u$. Suppose $1\le p,q\le\infty$, $T\ge 1$, and $\alpha\le\min\left\{\frac{1}{2},\frac{2}{n}\right\}$. Suppose $s>s_0$, where s_0 is picked appropriately to make the α-modulation space a multiplication algebra [4], k be a positive integer and there exists a constant c_k that is depended on k only such that

$$||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\le\frac{c_k}{T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)}T^{\frac{1}{2k}}},$$

and

$$||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\le\frac{c_k}{T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)}T^{1+\frac{1}{2k}}}.$$

Then the nonlinear Klein-Gordon equation has a unique solution u in $C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))$.

2. Preliminaries

Now we recall the definition of the α-modulation spaces. Let $0\le\alpha<1$, and $c<1$ and $C>1$ be two positive numbers which relate to the space dimension n. Suppose $\{\eta^\alpha_k\}_{k\in\mathbb{Z}^n}$ be a sequence of Schwartz functions that satisfies the following:

$$\begin{cases} |\eta^\alpha_k(\xi)|\succeq 1, \text{ if } \left|\xi-\langle k\rangle^{\frac{\alpha}{1-\alpha}}k\right|<c\langle k\rangle^{\frac{\alpha}{1-\alpha}}, \\ \operatorname{supp}\eta^\alpha_k\subset\left\{\xi\in\mathbb{R}^n:\left|\xi-\langle k\rangle^{\frac{\alpha}{1-\alpha}}k\right|<C\langle k\rangle^{\frac{\alpha}{1-\alpha}}\right\}, \\ \sum_{k\in\mathbb{Z}^n}\eta^\alpha_k(\xi)\equiv 1, \text{ for all } \xi\in\mathbb{R}^n, \\ \langle k\rangle^{\frac{\alpha|\delta|}{1-\alpha}}\left|D^\delta\eta^\alpha_k(\xi)\right|\preceq 1, \text{ for all } \xi\in\mathbb{R}^n \text{ and all multi-index } \delta, \end{cases} \tag{3}$$

where $\langle k\rangle=(1+|k|^2)^{\frac{1}{2}}$. The standard construct for a function that satisfies conditions (3) is to let ρ be a smooth radial bump function supported on the open ball of radius 2 centered at the origin that satisfies $\rho(\xi)=1$ when $|\xi|<1$ and $\rho(\xi)=0$ when $|\xi|\ge 2$. For any $k\in\mathbb{Z}^n$ define ρ^α_k by

$$\rho^\alpha_k(\xi)=\rho\left(\frac{\xi-\langle k\rangle^{\frac{\alpha}{1-\alpha}}k}{C\langle k\rangle^{\frac{\alpha}{1-\alpha}}}\right).$$

Now define η^α_k by

$$\eta^\alpha_k(\xi)=\rho(\xi)\left(\sum_{l\in\mathbb{Z}^n}\rho^\alpha_l(\xi)\right)^{-1}.$$

This η^α_k will satisfy conditions (3).

For $\{\eta^\alpha_k\}_{k=0}^\infty$ be a sequence of functions that satisfies conditions (3). Define $\Box^\alpha_k$ by

$$\Box^\alpha_k=\mathscr{F}^{-1}\eta^\alpha_k\mathscr{F}.$$

For $0<p,q\le\infty$, $s\in\mathbb{R}$, and $\alpha\in[0,1)$ define the norm $||\cdot||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}$ by

$$||f||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}=\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{\frac{sq}{1-\alpha}}||\Box^\alpha_k f||^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}}.$$

We now define the α-modulation space $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ as the set of all $f\in\mathscr{S}'$ such that $||f||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}<\infty$. See [5–7] for a way to define the α-modulation in a continuous way and the various properties. See [4,6,8,9] for additional properties of the α-modulation space, and its relation to the modulation space and the Besov space.

One property to note is the α-modulation space is multiplication algebra with the appropriate conditions on p and q [4]. This property will be used later.

Finally, to prove the main theorems, we need to establish the following lemma.

Lemma 1. *. Let $t \geq 1$ and $\square_k^\alpha$ be defined as above. Suppose there exists an $N > 0$ such that*

$$\left|\left|\square_k^\alpha e^{it(I-\Delta)^{\frac{\beta}{2}}} f\right|\right|_{L^1} \preceq t^{b_1} \left|\left|f\right|\right|_{L^1}, \tag{4}$$

if $|k| < N$ and

$$\left|\left|\square_k^\alpha e^{it(I-\Delta)^{\frac{\beta}{2}}} f\right|\right|_{L^1} \preceq t^{b_2} \langle k\rangle^d \left|\left|f\right|\right|_{L^1}, \tag{5}$$

if $|k| \geq N$, where $b_1 \geq b_2 \geq 0$ and d is a real number, then

$$\left|\left|e^{it(I-\Delta)^{\frac{\beta}{2}}} f\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \preceq t^{2b_1\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left|f\right|\right|_{M^{s-\gamma,\alpha}_{p,q}(\mathbb{R}^n)} + t^{2b_2\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left|f\right|\right|_{M^{s+\nu,\alpha}_{p,q}(\mathbb{R}^n)},$$

where $\gamma \geq 0$ and β is defined as

$$\nu = 2d(1-\alpha)\left|\frac{1}{p}-\frac{1}{2}\right|.$$

Proof. Follows from the same argument as [1] and [3]. $\square$

3. Proof of Theorem 1

In view of lemma 1 we need the following two proposition to proof theorem 1.

Proposition 1. *For $\beta > 1$ and $|k| = 0$, we have the following estimate:*

$$\left|\left|\mathscr{F}^{-1}\left(\eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}\right)\right|\right|_{L^1} \preceq t^{\frac{n}{2}}.$$

Proof. Let $L = \frac{n+1}{2}$ if n is odd and $L = \frac{n+2}{2}$ if n is even. First note that

$$\left|\left|\mathscr{F}^{-1}\left(\eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}\right)\right|\right|_{L^1} \preceq \int_{|x|\leq t}\left|\int_{\mathbb{R}^n} \eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx + \int_{|x|>t}\left|\int_{\mathbb{R}^n} \eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx.$$

For the first integral by Schwartz's inequality and Plancherel's Theorem we have

$$\int_{|x|\leq t}\left|\int_{\mathbb{R}^n} \eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx \preceq \left(\int_{|x|\leq t} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n}\left(\eta_0^k(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi}\right)^2 d\xi\right)^{\frac{1}{2}} \preceq t^{\frac{n}{2}} \left|\left|\eta_0^k(\xi)\right|\right|_{L^2} \preceq t^{\frac{n}{2}}.$$

For the second integral define $E_t = \{x \in \mathbb{R}^n : |x| > t\}$. For $i, j \in \{1, 2, \cdots, n\}$ define $E_{t,i} = \{x \in E_t : |x_i| \geq |x_j| \text{ for all } j \neq i\}$.

Now by integration-by-parts and a calculation

$$\begin{aligned}
\int_{|x|>t}\left|\int_{\mathbb{R}^n} \eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx &\preceq \sum_{i=1}^n \int_{E_{t,i}}\left|\int_{\mathbb{R}^n} \eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx \\
&\preceq \sum_{i=1}^n \int_{E_{t,i}} \frac{1}{|x|^L}\left|\int_{\mathbb{R}^n} \partial_{\xi_i}^L\left(\eta_0^\alpha(\xi) e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}\right) e^{ix\xi} d\xi\right| dx \\
&\preceq t^L \int_{|x|\leq t} \frac{1}{|x|^L}\left|\int_{\mathbb{R}^n} \sum_{\delta=1}^L \partial_{\xi_i}^{L-\delta} \eta_0^\alpha(\xi) |\xi|^{\delta\beta-\delta} e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx \\
&\preceq t^L \int_{|x|\leq t} \frac{1}{|x|^L}\left|\int_{\mathbb{R}^n} \eta_0^\alpha(\xi) |\xi|^{L\beta-L} e^{it(1+|\xi|^2)^{\frac{\beta}{2}}} e^{ix\xi} d\xi\right| dx.
\end{aligned}$$

In either case of n being odd or even, it follows that

$$2L(1-\beta)=(n+1)(1-\beta)<\frac{n+1}{2}<\frac{n}{2}+1.$$

Thus by Schwartz's inequality and noting that η_0^α has compact support it follows that

$$t^L\int_{|x|\leq t}\frac{1}{|x|^L}\left|\int_{\mathbb{R}^n}\eta_0^\alpha(\xi)|\xi|^{L\beta-L}e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}e^{ix\xi}d\xi\right|dx\preceq t^L\left(\int_{|x|>t}\frac{dx}{|x|^{2L}}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\eta_0^\alpha(\xi)|^2|\xi|^{2L(\beta-1)}d\xi\right)^{\frac{1}{2}}\preceq t^{\frac{n}{2}}.$$

This completes the proof. □

Proposition 2. *For $|k|\neq 0$ and $t>1$. If $\beta\geq 1$, then we have the following estimate:*

$$\left\|\mathscr{F}^{-1}\left(\eta_0^\alpha(\xi)e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}\right)\right\|_{L^1}\preceq t^{\frac{n}{2}}\langle k\rangle^{\frac{n(\beta-2+2\alpha)}{2(1-\alpha)}}.$$

Proof. Suppose $|k|\neq 0$. First making the substitutions of $\xi=\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi'+k)$ followed by $x=\frac{x'}{\langle k\rangle^{\frac{\alpha}{1-\alpha}}}$ to get

$$\begin{aligned}\left\|\mathscr{F}^{-1}\left(\eta_0^\alpha(\xi)e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}\right)\right\|_{L^1}&=\left\|\mathscr{F}^{-1}\left(\eta_k^\alpha(\xi)e^{it(1+|\xi|^2)^{\frac{\beta}{2}}}\right)\right\|_{L^1}\\&=\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k)\right)e^{it(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}}}e^{ix(\xi+k)}d\xi\right|dx\\&=\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\eta_k^\alpha\left(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k)\right)e^{i\left(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}}+x\xi\right)}d\xi\right|dx.\end{aligned}$$

Define Φ as

$$\begin{aligned}\Phi&=t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}}+x\xi\\&=\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}t\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta}{2}}+x\xi.\end{aligned}$$

Then $\frac{\partial\Phi}{\partial\xi_i}=\beta t\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi_i+k_i)\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-2}{2}}+x_i$, and

$$\frac{\partial^2\Phi}{\partial\xi_i\partial\xi_j}=\begin{cases}\beta(\beta-2)t\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi_i+k_i)(\xi_j+k_j)\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-4}{2}}, & \text{if } i\neq j,\\ \beta(\beta-2)t\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi_i+k_i)^2\left(\langle k\rangle^{-\frac{2\beta}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-4}{2}}+\beta t\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}\left(\langle k\rangle^{\frac{-2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-2}{2}}, & \text{if } i=j.\end{cases}$$

Also note $\frac{\partial\Phi}{\partial\xi_i}=0$ when

$$x_i=-\beta t\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi_i+k_i)\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-2}{2}},$$

or equivalently

$$x=-\beta t\langle k\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi+k)\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-2}{2}}.$$

Now for the case of $n=2$ we have

$$\left|\det(D_{\xi_i}D_{\xi_j}\Phi)_{i,j=1}^2\right| = \left|\left(\beta(\beta-2)t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi_1+k_1)^2\left(\left\langle k\right\rangle^{-\frac{2\beta}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-4}{2}}+\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}\left(\left\langle k\right\rangle^{\frac{-2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-2}{2}}\right)\right.$$
$$\times\left(\beta(\beta-2)t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}(\xi_2+k_2)^2\left(\left\langle k\right\rangle^{-\frac{2\beta}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-4}{2}}+\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}\left(\left\langle k\right\rangle^{\frac{-2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{\frac{\beta-2}{2}}\right)$$
$$\left.-\beta^2(\beta-2)^2t^2\left\langle k\right\rangle^{\frac{2\beta\alpha}{1-\alpha}}(\xi_1+k_1)^2(\xi_2-k_2)^2\left(\left\langle k\right\rangle^{-\frac{2\alpha}{1-\alpha}}-|\xi+k|^2\right)^{\beta-4}\right|$$
$$=\beta^2t^2\left\langle k\right\rangle^{\frac{2\beta\alpha}{1-\alpha}}\left|\left\langle k\right\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right|^{\frac{2\beta-4}{2}}\times\left((\beta-2)\left(\left\langle k\right\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{-1}|\xi+k|^2+1\right).$$

Then $\left|\det(D_{\xi_i}D_{\xi_j}\Phi)_{i,j=1}^2\right|=0$ only if $(\beta-2)\left(\left\langle k\right\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{-1}|\xi+k|^2+1=0$, which only happens when $\beta=1-\left\langle k\right\rangle^{-\frac{2\alpha}{1-\alpha}}|\xi+k|^{-2}<1$.

Thus when $\beta\geq 1$ and when $k\neq 0$, $\left|\det(D_{\xi_i}D_{\xi_j}\Phi)_{i,j=1}^2\right|\neq 0$. Also note that when $k\neq 0$

$$\left|\det(D_{\xi_i}D_{\xi_j}\Phi)_{i,j=1}^2\right|\sim\beta^2t^2\left\langle k\right\rangle^{\frac{2\beta\alpha}{1-\alpha}}|\xi+k|^{2\beta-4}\left(\alpha-1+\frac{1}{1+\left\langle k\right\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2}\right)$$
$$\geq\left(\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|\xi+k|^{\beta-2}\right)^2\left(\alpha-1+\frac{1}{1+\left\langle k\right\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2}\right)$$
$$\geq\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2}.$$

these calculation can be extended for $n\geq 3$.

Define $C_i(k)$ and $D_i(k)$ as

$$C_i(k)=\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}(|k_i|+C)\left(\sum_{j=1}^n(|k_j|+C)^2\right)^{\frac{\beta-2}{2}},$$
$$D_i(k)=\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}(|k_i|-C)\left(\sum_{j=1}^n(|k_j|-C)^2\right)^{\frac{\beta-2}{2}}.$$

Define the intervals F_i as the set of all $x_i\in\mathbb{R}$, such that,

$$D_i(k)-t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2}<|x_i|<C_i(k)+t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2},$$

$G_{i,j}$ to be the set of all $x_i\in\mathbb{R}$ such that

$$C_i(k)+t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2}+j-1<|x_i|\leq C_i(k)+t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2}+j,$$

and $H_{i,j}$ to be the set of all $x_i\in\mathbb{R}$ such that

$$D_i(k)-t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2}-j<|x_i|\leq D_i(k)+t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2}-j+1.$$

Since $|x|=\beta t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|\xi+k|\left|\left\langle k\right\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right||^{\frac{\beta-2}{2}}$, then it follows that $x_i\in F_i$. It also follows that

$$\text{length}(F_i)\preceq t\left\langle k\right\rangle^{\frac{\beta\alpha}{1-\alpha}}|k|^{\beta-2},\text{ and}$$
$$\text{length}(G_{i,j})=\text{length}(H_{i,j})=1.$$

Now define $K_{i,j} = G_{i,j} \cup H_{i,j}$, then it follows that $\chi_{F_i}(x_i) + \sum_{j=1}^{\infty} \chi_{K_{i,j}}(x_i) = 1$. Thus we have

$$\begin{aligned}
\left\|\mathscr{F}^{-1}(\eta_k^\alpha(\xi)e^{it(1+|\xi|^2)^{\frac{\beta}{2}}})\right\|_{L^1} &\preceq \int_{\mathbb{R}^n} \prod_{i=1}^n \chi_{F_i}(x_i) \left| \int_{\mathbb{R}^n} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right) e^{it(1+\langle k \rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}} + ix\xi} d\xi \right| dx \\
&+ \sum_{j_*=1}^n \sum_{I_l} \int_{\mathbb{R}^n} \mathscr{A}_{I_l}(x) \left| \int_{\mathbb{R}^n} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right) e^{it(1+\langle k \rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}} + ix\xi} d\xi \right| dx \\
&+ \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \int_{\mathbb{R}^n} \prod_{i=1}^n \chi_{K_{i,j_i}}(x_i) \left| \int_{\mathbb{R}^n} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right) e^{it(1+\langle k \rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}} + ix\xi} d\xi \right| dx \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $\mathscr{A}_{I_l}$ is the product of characteristic functions $\chi_{F_i}(x_i)$ and $\chi_{K_{i,j_*}}(x_i)$ where there is at least one $\chi_{F_i}(x_i)$ and at least one $\chi_{K_{i,j_*}}(x_i)$.

For I_1, with $\xi \in \operatorname{supp} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right)$ and Van der Corput Lemma, see proposition 2.6.4 in [10], we have

$$\begin{aligned}
I_1 &\preceq \left(t \langle k \rangle^{\frac{\beta\alpha}{1-\alpha}} |k|^{\beta-2} \right)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \prod_{i=1}^n \chi_{F_i}(x_i) dx \preceq \left(t \langle k \rangle^{\frac{\beta\alpha}{1-\alpha}} |k|^{\beta-2} \right)^{-\frac{n}{2}} \left(t \langle k \rangle^{\frac{\beta\alpha}{1-\alpha}} |k|^{\beta-2} \right)^{n} \\
&= t^{\frac{n}{2}} \langle k \rangle^{\frac{n\beta\alpha}{2(1-\alpha)}} |k|^{\frac{n(\beta-2)}{2}} \preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\beta-2+2\alpha)}{2(1-\alpha)}}.
\end{aligned}$$

For additional details of Van der Corput lemma see [11,12].

Now note for $x \in K_{i,j}$ and $\xi \in \operatorname{supp} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right)$ we have

$$\begin{aligned}
\frac{\partial}{\partial \xi_l} \left(\frac{\eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right)}{\frac{\partial}{\partial \xi_i} \Phi} \right) &= \frac{\frac{\partial \Phi}{\partial \xi_i} \frac{\partial}{\partial \xi_l} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right) - \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right) \frac{\partial^2 \Phi}{\partial \xi_l \partial \xi_i}}{\left(\frac{\partial \Phi}{\partial \xi_i} \right)^2} \\
&\preceq \frac{\beta t \langle k \rangle^{\frac{\beta\alpha}{1-\alpha}} |\xi + k|^{\beta-2} (\xi_i + k_i) + x_i - \frac{\partial^2 \Phi}{\partial \xi_l \partial \xi_i}}{\left(\beta t \langle k \rangle^{\frac{\beta\alpha}{1-\alpha}} (\xi_i + k_i) \left| \langle k \rangle^{-\frac{2\alpha}{1-\alpha}} + |\xi + k|^2 \right|^{\frac{\beta-2}{2}} + x_i \right)^2} \\
&= O \left(\frac{1}{j + \sqrt{t} \langle k \rangle^{\frac{\beta\alpha}{2(1-\alpha)}} |k|^{\frac{\beta-2}{2}}} + \frac{t \langle k \rangle^{\frac{\beta\alpha}{1-\alpha}} |k|^{\beta-2}}{\left(j + \sqrt{t} \langle k \rangle^{\frac{\beta\alpha}{2(1-\alpha)}} |k|^{\frac{\beta-2}{2}} \right)^2} \right).
\end{aligned}$$

Thus, using integration-by-parts twice on each variable $\xi_1, \cdots, \xi_n$ we have

$$\begin{aligned}
I_3 &\preceq \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \frac{t^n \langle k \rangle^{\frac{n\beta\alpha}{1-\alpha}} |k|^{n\beta-2n}}{\prod_{i=1}^n \left(j_i + \sqrt{t} \langle k \rangle^{\frac{\beta\alpha}{2(1-\alpha)}} |k|^{\frac{\beta-2}{2}} \right)^2} \int_{\mathbb{R}^n} \prod_{i=1}^n \chi_{K_{i,j_i}}(x_i) dx \\
&\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\beta\alpha}{2(1-\alpha)}} |k|^{\frac{n(\beta-2)}{2}} \\
&\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\beta-2+2\alpha)}{2(1-\alpha)}}.
\end{aligned}$$

When $\xi \in \operatorname{supp} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right)$, then I_2 is the sum of integrals of the form

$$\sum_{j_{l+1}=1}^n \cdots \sum_{j_n=1}^n \int_{\mathbb{R}^n} \prod_{i_0=1}^l \chi_{F_{i_0}}(x_{i_0}) \prod_{i_0=l+1}^n \chi_{K_{i_0,j_{i_0}}}(x_{i_0}) \left| \int_{\mathbb{R}^n} \eta_k^\alpha \left(\langle k \rangle^{\frac{\alpha}{1-\alpha}} (\xi + k) \right) e^{it(1+\langle k \rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{\beta}{2}} + x\xi} d\xi \right| dx.$$

So doing integration-by-parts twice on the variables $\xi_{l+1}, \cdots, \xi_n$, the above integral is bounded by

$$\sum_{j_{l+1}=1}^{n} \cdots \sum_{j_n=1}^{n} \frac{t^{n-l} \langle k \rangle^{\frac{(n-l)\beta\alpha}{1-\alpha}} |k|^{(n-l)\beta-2n}}{\prod_{i=l+1}^{n} \left(j_i + \sqrt{t} \langle k \rangle^{\frac{\beta\alpha}{2(1-\alpha)}} |k|^{\frac{\beta-2}{2}} \right)^2} \times \int_{\mathbb{R}^n} \prod_{i_0=1}^{l} \chi_{F_{i_0}}(x_{i_0}) \prod_{i_0=l+1}^{n} \chi_{K_{i_0,j_{i_0}}}(x_{i_0}) dx$$
$$\preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n\beta\alpha}{2(1-\alpha)}} |k|^{\frac{n(\beta-2)}{2}} \preceq t^{\frac{n}{2}} \langle k \rangle^{\frac{n(\beta-2+2\alpha)}{2(1-\alpha)}}.$$

This completes the proof. □

4. Proof of Theorem 2

We now present the proof of theorem 2, which is proved through the following two propositions and lemma 1.

Proposition 3. *For $1 \le p, q \le \infty$, $t \ge 1$ and $|k| = 0$, then the following estimate holds*

$$||\Box_0^\alpha \Theta_K g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{L^p(\mathbb{R}^n)}.$$

Proof. Suppose $|k| = 0$. Let L be defined by the same as in proposition 1. then by Bernstein's Lemma we have

$$||\Box_0^\alpha \Theta_K g||_{L^1(\mathbb{R}^n)} \preceq \left|\left| \eta_0^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \times \sum_{|\delta|=L} \left|\left| D^\delta \left(\eta_0^k(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right) \right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} ||g||_{L^1(\mathbb{R}^n)}.$$

For the first norm we have

$$\left|\left| \eta_0^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} = \left|\left| t\, \eta_0^\alpha(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{t(1+|\xi|^2)^{\frac{1}{2}}} \right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}}$$
$$\preceq t^{1-\frac{n}{2L}}.$$

For the second norm, define $h(|\xi|) = \frac{\sin((1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}$. Note that h is a C^∞ function. Also, by a calculation we have $\lim_{|\xi|\to\infty} |D^\delta h(|\xi|)| = 0$, for all multi-indices δ. Noticing that $th(t|\xi|) = \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}$, then when $|\delta| = L$ we have

$$\left| D^\delta \left(\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right) \right| \preceq t^{L+1} \sup_{\xi\in\mathbb{R}^n} \left| D^\delta h(|\xi|) \right| \preceq t^{L+1}.$$

Now it follows that

$$\sum_{|\delta|=L} \left|\left| D^\delta \left(\eta_0^k(\xi) \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \right) \right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} \preceq \left(t^{L+1} \right)^{\frac{n}{2L}} = t^{\frac{n}{2}} t^{\frac{n}{2L}}.$$

Therefore we have

$$||\Box_0^\alpha \Theta_K(t) g||_{L^1(\mathbb{R}^n)} \preceq t^{\frac{n}{2}} t^{\frac{n}{2L}} t^{1-\frac{n}{2L}} ||g||_{L^1(\mathbb{R}^n)} = t^{\frac{n+2}{2}} ||g||_{L^1(\mathbb{R}^n)}.$$

By Plancherel's Theorem it follows that

$$||\Box_0^\alpha \Theta_K(t) g||_{L^2(\mathbb{R}^n)} \preceq \left|\left| \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}} \hat{g} \right|\right|_{L^2(\mathbb{R}^n)} = \left|\left| t \frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{t(1+|\xi|^2)^{\frac{1}{2}}} \hat{g} \right|\right|_{L^2(\mathbb{R}^n)} \preceq t\, ||\hat{g}||_{L^2(\mathbb{R}^n)} = t\, ||g||_{L^2(\mathbb{R}^n)}.$$

Now by Riesz-Thorin Interpolation and a duality argument it follows that

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{L^p(\mathbb{R}^n)}.$$

This completes the proof. □

Proposition 4. *For $1 \le p, q \le \infty$, $t \ge 1$, and $k \ne 0$, then the following estimate holds*

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{L^p(\mathbb{R}^n)}.$$

Proof. Suppose $|k| \ne 0$. Again define L to be defined as in the proof of theorem 1. Then by the usual substitutions and Bernstein's lemma we have

$$||\Box_k^\alpha \Theta_K(t)g||_{L^1(\mathbb{R}^n)} \preceq \left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \sum_{|\delta|=L}\left|\left|D^\delta \frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))\sin(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}})}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}}.$$

For the first norm, noting that for large enough k we have $(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2)^{-\frac{1}{2}} \preceq \langle k\rangle^{-1}$ it follows that

$$\begin{aligned}\left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} &= \langle k\rangle^{-\frac{\alpha}{1-\alpha}+\frac{n\alpha}{2L(1-\alpha)}} \times \left|\left|\frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))}{(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2L}} \\ &\preceq \langle k\rangle^{-\frac{\alpha}{1-\alpha}+\frac{n\alpha}{2L(1-\alpha)}}\langle k\rangle^{-1+\frac{n}{2L}} \\ &= \langle k\rangle^{\frac{n-2L}{2L(1-\alpha)}}.\end{aligned}$$

For the second norm, $D^\delta \sin(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}})$ produces t^L and $\langle k\rangle^{\frac{L\alpha}{1-\alpha}}$ factors when $|\delta| = L$. Also, after taking multiple derivatives we have the remaining factors of the form $(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{-\frac{j}{2}}$ for some positive integer j which again for large enough k we have

$$\begin{aligned}(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{-\frac{j}{2}} &\preceq \langle k\rangle^{-\frac{j\alpha}{1-\alpha}}\left(\langle k\rangle^{-\frac{2\alpha}{1-\alpha}}+|\xi+k|^2\right)^{-\frac{j}{2}} \\ &\preceq \langle k\rangle^{-\frac{\alpha}{1-\alpha}}\langle k\rangle^{-1} = \langle k\rangle^{-\frac{1}{1-\alpha}}.\end{aligned}$$

Thus we have

$$\sum_{|\delta|=L}\left|\left|D^\delta \frac{\eta_k^\alpha(\langle k\rangle^{\frac{\alpha}{1-\alpha}}(\xi+k))\sin(t(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}})}{(1+\langle k\rangle^{\frac{2\alpha}{1-\alpha}}|\xi+k|^2)^{\frac{1}{2}}}\right|\right|_{L^2(\mathbb{R}^n)}^{\frac{n}{2L}} \preceq t^{\frac{n}{2}}\langle k\rangle^{\frac{n\alpha}{2(1-\alpha)}}\langle k\rangle^{-\frac{n\alpha}{2L(1-\alpha)}}\langle k\rangle^{-\frac{n}{2L}} = t^{\frac{n}{2}}\langle k\rangle^{\frac{n\alpha L-n}{2L(1-\alpha)}}.$$

Then it follows that

$$||\Box_k^\alpha \Theta_K(t)g||_{L^1(\mathbb{R}^n)} \preceq t^{\frac{n}{2}}\langle k\rangle^{\frac{n-2L}{2L(1-\alpha)}}\langle k\rangle^{\frac{n\alpha L-n}{2L(1-\alpha)}} = t^{\frac{n}{2}}\langle k\rangle^{\frac{\alpha n-2}{2(1-\alpha)}}.$$

Using Plancherel's theorem we have

$$\begin{aligned}||\Box_k^\alpha \Theta_K(t)g||_{L^2(\mathbb{R}^n)} &= \left|\left|\eta_k^\alpha(\xi)\frac{\sin(t(1+|\xi|^2)^{\frac{1}{2}})}{(1+|\xi|^2)^{\frac{1}{2}}}\hat{g}\right|\right|_{L^2(\mathbb{R}^n)} \\ &\preceq ||\hat{g}||_{L^2(\mathbb{R}^n)} = ||g||_{L^2(\mathbb{R}^n)}.\end{aligned}$$

By Riesz-Thorin and a duality argument it follows that for $1 \le p \le \infty$

$$||\Box_k^\alpha \Theta_K(t)g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|}||g||_{L^p(\mathbb{R}^n)},$$

and this finishes the proof. □

Proof. To finish the *proof of theorem* 2 we first notice that by almost orthogonality we have

$$\begin{aligned}\Box_k^\alpha \Theta_K(t) g &= \sum_{|l|\leq \gamma_{C,k}} \Box_{k+l}^\alpha \Box_k^\alpha \Theta_K(t) g \\ &= \sum_{|l|\leq \gamma_{C,k}} \Box_{k+l}^\alpha \Theta_K(t) \Box_k^\alpha g,\end{aligned}$$

where $\gamma_{C,k}$ is a constant dependent on C and k. Then by proposition 3 we have for small values of k

$$||\Box_k^\alpha \Theta_K(t) g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||\Box_k^\alpha g||_{L^p(\mathbb{R}^n)},$$

and by proposition 4 we have for large values of k

$$||\Box_k^\alpha \Theta_K(t) g||_{L^p(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \langle k\rangle^{\frac{\alpha n-2}{1-\alpha}\left|\frac{1}{p}-\frac{1}{2}\right|} ||\Box_k^\alpha g||_{L^p(\mathbb{R}^n)}.$$

By definition of the α-Modulation norm and following the same argument as [1] and [3] we have

$$||\Theta_K(t) g||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M_{p,q}^{s-\gamma,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M_{p,q}^{s+\beta_1(\alpha),\alpha}(\mathbb{R}^n)}.$$

This completes the proof. □

5. Application to Klein-Gordon Type Equations

Corollary 1. *Let $1 \leq p, q \leq \infty$, $t \geq 1$, and $u(t,x)$ be the solution to the Cauchy Problem for the Klein-Gordon Equation*

$$\begin{cases} \partial_{tt} u(t,x) + u(t,x) - \Delta u(t,x) = 0, & \textit{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = f(x), & \textit{for } x \in \mathbb{R}^n, \\ \partial_t u(0,x) = g(x), & \textit{for } x \in \mathbb{R}^n, \end{cases}$$

then we have the followings estimate

$$\begin{aligned}||u||_{M_{p,q}^{s,\alpha}(\mathbb{R}^n)} \preceq{} & t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M_{p,q}^{s-\gamma_1,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||f||_{M_{p,q}^{s+\nu_1(1,\alpha),\alpha}(\mathbb{R}^n)} \\ &+ t^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} ||g||_{M_{p,q}^{s-\gamma_2,\alpha}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||g||_{M_{p,q}^{s+\nu_2(\alpha),\alpha}(\mathbb{R}^n)},\end{aligned}$$

where γ_1 and γ_2 are positive real numbers, $\nu_1(1,\alpha)$ and $\nu_2(\alpha)$ are defined as in equation (1) and (2) respectively.

The formal solution to this equation is given by

$$u(t,x) = \cos\left(t(I-\Delta)^{\frac{1}{2}}\right) u_0(x) + \Theta_K(t) u_1(x),$$

where $\cos\left(t(I-\Delta)^{\frac{1}{2}}\right)$ is the Fourier multiplier with symbol $\cos\left(t(1+|\xi|^2)^{\frac{1}{2}}\right)$. The Fourier multiplier $\cos\left(t(I-\Delta)^{\frac{1}{2}}\right)$, or equivalent $e^{it(I-\Delta)^{\frac{1}{2}}}$, estimate is given by theorem 1. Then with theorem 2 corollary 1 follows.

Now we close the paper with two applications: the (half) Klein-Gordon equation and the nonlinear Klein-Gordon equation. Define the function space

$$C\left([0,T], M_{p,q}^{s,\alpha}\right) = \left\{u(t,x) : ||u||_{C\left([0,T],M_{p,q}^{s,\alpha}\right)} < \infty\right\},$$

where $||u||_{C\left([0,T],M^{s,\alpha}_{p,q}\right)} = \sup_{0\leq t\leq T} ||u(t,\cdot)||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}$.

Theorem 4. *Let $1 \leq p, q \leq \infty$, $s > s_0$ where s_0 is defined appropriately to make the α-modulation space a multiplication algebra [4], and $T \geq 1$. Suppose k is a positive integer and there is positive constant c_k dependent only on k such that*

$$||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{c_k}{T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)} T^{\frac{1}{2k}}}.$$

Suppose $1 \leq \beta \leq 2(1-\alpha)$, then the nonlinear Klein-Gordon type equation of the form

$$\begin{cases} i\partial_t u - (I-\Delta)^{\frac{\beta}{2}} u + F(u) = 0, & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & \text{for } x \in \mathbb{R}^n, \end{cases}$$

where $F(u) = |u|^{2k}u$ has a unique solution $u \in C([0,T], M^{s,\alpha}_{p,q})$.

Proof. Consider the mapping

$$\mathscr{T}_K u = e^{it(I-\Delta)^{\frac{\beta}{2}}} u_0 - \int_0^t e^{i(t-\tau)(I-\Delta)^{\frac{\beta}{2}}} F(u(\tau,\cdot))d\tau.$$

Let C_j where $j = 1, 2, 3$ denote some positive constants that are independent of all essential variables. By Theorem 1 we have

$$\left|\left|e^{it(I-\Delta)^{\frac{\beta}{2}}} u_0\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq C_1(1+t)^{n\left|\frac{1}{p}-\frac{1}{2}\right|} ||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.$$

Since the α-modulation space is a multiplication algebra when $s > s_0$ there is a constant $A_{2k+1} > 0$ such that

$$\left|\left||u(t,\cdot)|^{2k+1}\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq A_{2k+1} ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.$$

Let $M_k = \max\{A_{2k}, A_{2k+1}\}$. Now for any $T \geq 1$ and $t \leq T$

$$\begin{aligned} \left|\left|\int_0^t e^{i(t-\tau)(I-\Delta)^{\frac{\beta}{2}}} F(u(\tau,\cdot))d\tau\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} &\leq C_1 \int_0^t (1+(t-\tau))^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left|\left||u(\tau,\cdot)|^{2k}u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} d\tau \\ &\leq C_2 M_k T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} \sup_{0\leq t\leq T} ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}. \end{aligned}$$

Thus it follows that

$$||\mathscr{T}_K u||_{C([0,T],M^{s,\alpha}_{p,q})} \preceq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T \sup_{0\leq t\leq T} ||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \right).$$

Let $\mathcal{L} = \frac{1}{\left(2C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\right)^{\frac{1}{2k}} (2k+1)^{\frac{1}{2k}}}$, and let $B_{\mathcal{L}}$ be the closed ball of radius $\mathcal{L}$ centered at the origin in the space of $C([0,T], M^{s,\alpha}_{p,q})$. Suppose that

$$||u_o||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{1}{(2k+1)^{\frac{1}{2k}} (2C_3)^{1+\frac{1}{2k}} T^{n\left|\frac{1}{p}-\frac{1}{2}\right|\left(1+\frac{1}{2k}\right)} T^{\frac{1}{2k}}}.$$

Thus it follows that

$$||\mathscr{T}_K u||_{C([0,T],M^{s,\alpha}_{p,q})} \leq C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|} \left(||u_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T\mathcal{L}^{2k+1} \right) \leq \mathcal{L},$$

and so $\mathscr{T}_K$ is a mapping from $B_{\mathcal{L}}$ into $B_{\mathcal{L}}$.

Then

$$\mathscr{T}_K u - \mathscr{T}_K v = e^{it(I-\Delta)^{\frac{\beta}{2}}}(u_0 - v_0) - \int_0^t e^{i(t-\tau)(I-\Delta)^{\frac{\beta}{2}}}\left(|u|^{2k}u - |v|^{2k}v\right)d\tau.$$

With the above and Lemma from [1] we have that

$$\begin{aligned}
||\mathscr{T}_K u - \mathscr{T}_K v||_{C([0,T],M^{s,\alpha}_{p,q})} &\le C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\left(||u_0 - v_0||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + \sup_{0\le t\le T}||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right)\\
&= C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\sup_{0\le t\le T}\left(||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right)\\
&\le C_3 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\sup_{0\le t\le T}||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}(2k+1)\mathcal{L}^{2k}\\
&\le \frac{1}{2}\sup_{0\le t\le T}||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} = \frac{1}{2}||u-v||_{C([0,T],M^{s,\alpha}_{p,q})}.
\end{aligned}$$

This show that $\mathscr{T}_K$ is a contraction map on $B_{\mathcal{L}}$. Thus, by the fixed point theorem we have a unique solution in $B_{\mathcal{L}}$. □

Now we present the *proof for Theorem 3*

Proof. Let C_j where $j = 1, 2, 3, 4$ are all essential constants. Define the map $\mathscr{T}_{KG}$ by

$$\mathscr{T}_{KG} u = \cos(t(I-\Delta)^{\frac{1}{2}})f_u(x) + \Theta_K(t)g_u(x) - \int_0^t \Theta_K(t-\tau)F(u(\tau,x))d\tau.$$

By the previous theorems and hypothesis we have

$$\left|\left|\cos(t(I-\Delta)^{\frac{1}{2}})f_u + \Theta_K(t)g_u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \le C_1\left(t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + t^{n\left|\frac{1}{p}-\frac{1}{2}\right|}(1+t)\,||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right).$$

Furthermore, we have

$$\begin{aligned}
\left|\left|\int_0^t \Theta_K(t-\tau)F(u(\tau,x))d\tau\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} &\le C_2\int_0^T\left((1+(t-\tau))^{n\left|\frac{1}{p}-\frac{1}{2}\right|} + (1+(t-\tau))^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}\right)\\
&\quad\times\left|\left||u|^{2k}u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}d\tau\\
&\le C_2 M_k\left(T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1} + T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\right)\left|\left||u|^{2k}u\right|\right|_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\\
&\le C_2 M_k T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}(1+T)\sup_{0\le t\le T}||u||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\\
&\le C_3 M_k T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\sup_{0\le t\le T}||u||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)},
\end{aligned}$$

Thus we have

$$\begin{aligned}
||\mathscr{T}_{KG}u||_{C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))} &\le C_4 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + C_4 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+1}||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\\
&\quad + C_4 T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\sup_{0\le t\le T}||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}.
\end{aligned}$$

Define $\mathcal{L}_{KG} = \frac{1}{(3C_4)^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}\left(T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\right)^{\frac{1}{2k}}}$, and $B_{\mathcal{L}_{KG}}$ be an open ball centered at the origin in $C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))$ with radius $\mathcal{L}_{KG}$. Suppose that the following estimates hold

$$||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{1}{(3C_4)^{1+\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}T^{n\left|\frac{1}{p}-\frac{1}{2}\right|(1+\frac{1}{2k})}T^{\frac{1}{k}}},$$

and

$$||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq \frac{1}{(3C_4)^{1+\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}T^{n\left|\frac{1}{p}-\frac{1}{2}\right|(1+\frac{1}{2k})}T^{\frac{1}{k}+1}}.$$

It follows that

$$\begin{aligned}||\mathscr{T}_{KG}u||_{C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))} &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\left(||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T\,||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T^2\sup_{0\leq t\leq T}||u(t,\cdot)||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right)\\ &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\frac{3}{(3C_4)^{1+\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}T^{n\left|\frac{1}{p}-\frac{1}{2}\right|(1+\frac{1}{2k})}T^{\frac{1}{k}}} = \frac{1}{(3C_4)^{\frac{1}{2k}}(2k+1)^{\frac{1}{2k}}\left(T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\right)^{\frac{1}{2k}}}.\end{aligned}$$

Therefore, $\mathscr{T}_{KG}: B_{\mathcal{L}_{KG}} \to B_{\mathcal{L}_{KG}}$. Furthermore, we have

$$\begin{aligned}\mathscr{T}_{KG}u - \mathscr{T}_{KG}v &= \cos(t(I-\Delta)^{\frac{1}{2}})(f_u(x)-f_v(x)) + \Theta_K(t)(g_u(x)-g_v(x))\\ &\quad - \int_0^t \Theta_K(t-\tau)(F(u(\tau,x))-F(v(\tau,x)))d\tau.\end{aligned}$$

Now using the hypothesis we have $||g_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} \leq ||f_u||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}$, we have

$$\begin{aligned}||\mathscr{T}_{KG}u - \mathscr{T}_{KG}||_{C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))} &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\Big(||f_u-f_v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T\,||g_u-g_v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\\ &\quad + T^2\sup_{0\leq t\leq T}||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\Big)\\ &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|}\left((2+t)\,||f_u-f_v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + T^2\sup_{0\leq t\leq T}||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right)\\ &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\sup_{0\leq t\leq T}\left(||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} + ||u-v||^{2k+1}_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\right)\\ &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\sup_{0\leq t\leq T}||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\,(2k+1)\mathcal{L}^{2k}_{KG}\\ &\leq C_4T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}\frac{2k+1}{3c_3(2k+1)T^{n\left|\frac{1}{p}-\frac{1}{2}\right|+2}}\sup_{0\leq t\leq T}||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)}\\ &\leq \frac{1}{3}\sup_{0\leq t\leq T}||u-v||_{M^{s,\alpha}_{p,q}(\mathbb{R}^n)} = \frac{1}{3}||u-v||_{C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))},\end{aligned}$$

therefore $\mathscr{T}_{KG}$ is a contraction map and by the fixed point theorem there exists a unique solution $u \in C([0,T]M^{s,\alpha}_{p,q}(\mathbb{R}^n))$. This completes the proof. □

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