Asian Research Journal of Mathematics

17(12): 63-70, 2021; Article no.ARJOM.79302
ISSN: 2456-477X

Decay for Solutions to a Plate Type Equation

Shuo Xu [a*] and Shikuan Mao [a]

[a] *School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China*.

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2021/v17i1230349

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/79302

Received 20 October 2021
Accepted 23 December 2021
Published 25 December 2021

Original Research Article

Abstract

In a similar spirit to the papers [1,2], existence and decay for a plate type equation is obtained. The result is a generalization of the work for the linear equation in the paper [1].

Keywords: Decay estimates; plate type equation; semilinear; memory term.

1 Preliminary

We study the following equation in $R^n \times [0,+\infty)$ ($n \geq 1$)

$$\begin{cases} \partial_t^2 \chi + (\Delta^2 + 1)\chi - \kappa *_t \chi = g(\partial_x^2 \chi, \partial_t \chi), \\ \chi(0) = \chi_0(x), \qquad \partial_t \chi(0) = \chi_1(x). \end{cases} \tag{P}$$

The memory term $\kappa *_t \chi$ is defined by

$$(\kappa *_t \chi) := \int_0^t \kappa(t-s)\chi(x,s)ds\,.$$

**Corresponding author: Email: 1206282196@qq.com;*

We assume the memory kernel $\kappa(t)$ satisfies the following assumption.

$$(a) \quad 0<\kappa \in C^2\left(\left[0,\infty\right)\right),$$
$$(b) \quad -C_1 \le \kappa'(t)/g(t) \le -C_2, \quad \left|\kappa''(t)/\kappa(t)\right| \le C_3 \text{ for t} \ge 0,$$
$$(c) \quad \int_0^\infty \kappa(s)\,ds \le 1,$$

with $C_j > 0\ (j=1,2,3)$ are constants.

And $g(\partial_x^2\chi, \partial_t\chi)$ satisfies the following assumption.

$$g(\lambda\partial_x^2\chi, \lambda\partial_t\chi) = \lambda^\alpha g(\partial_x^2\chi, \partial_t\chi), \forall \lambda > 0,$$

here α is an integer satisfying $\alpha > \alpha_n$ with $\alpha_n := \begin{cases} 5-n, & n \le 3, \\ 1+\dfrac{2}{n}, & n \ge 4. \end{cases}$

For $k \in Z^+$, we denote

$$\omega(k,n) = 2k + \left[\frac{n+1}{2}\right], n \ge 1.$$

Theorem1.1 Let $s \in Z^+$, $s \ge \max\{n+1,3\}$. Suppose

$\chi_0 \in H^{s+2} \cap L^1$ and $\chi_1 \in H^s \cap L^1$. Put

$$E_0 := \|\chi_0\|_{H^{s+2}} + \|\chi_1\|_{H^s} + \|\chi_0\|_{L^1} + \|\chi_1\|_{L^1}.$$

Then there exists uniquely a solution $\chi \in C^0([0,\infty); H^{s+2}) \cap C^1([0,\infty); H^s)$ of (P) satisfying the following estimates:

$$\left\|\partial_x^{k+2}\chi(t)\right\|_{H^{s-\omega(k,n)}} \le CE_0(1+t)^{-(\frac{n}{8}+\frac{k}{4})},$$
$$\left\|\partial_x^k\partial_t\chi(t)\right\|_{H^{s-\omega(k,n)}} \le CE_0(1+t)^{-(\frac{n}{8}+\frac{k}{4})}.$$

Here $k \ge 0$ satisfying $\omega(k,n) \le s$.

We recall some related work. Da Luz-Charão (see [3]) studied the following dissipative plate equation in a bounded domain in R^n $(1 \le n \le 5)$

$$(1-\Delta)\partial_t^2\chi + \Delta^2\chi + \partial_t\chi = g(\chi).$$

Here $\partial_t\chi$ is the linear dissipative term. Sugitani-Kawashima (see [4]) studied this problem in R^n and extend the results to general n. Subsequently, Liu-Kawashima (see [5,6]) studied a more complex equation

$$(1-\Delta)\partial_t^2\chi+\sum_{i,j=1}^{n}a^{ij}(\partial_x^2\chi)_{x_ix_j}+\partial_t\chi=0.$$

In [7], Liu-Kawashima studied the following memory type equation

$$\partial_t^2\chi+(\Delta^2+1)\chi+\kappa *_t \Delta\chi=g(\chi).$$

Liu (see [2], also [8] for related results) further studied the following Cauchy problem

$$(1-\Delta)\partial_t^2\chi+(\Delta^2+1)\chi+\kappa *_t \Delta\chi=g(\chi,\partial_t\chi,\nabla\chi).$$

Mao-Liu (see [9]) generalized the results of plate-type equation (see [2,7]) with memory to higher order equations. They studied fractional order of derivatives. They also (see [10]) studied equations of variable coefficients.

In these papers, the memory term under consideration is $\kappa *_t \Delta\chi$. Recently, Liu-Ueda (see [1]) studied a type of linear plate equation with some different memory term $\kappa *_t \chi$. They obtained some decay estimates and asymptotical behavior for solutions under suitable assumption.

Similar results also holds for Timoshenko system (see [11,12]) and hyperbolic-elliptic system (see [13]). For more related results, we refer to [14,15,16,17,5,18].

In section 2, we will prove Theorem 1.1, which extends the result in [1] to the case of semi-linear perturbations.

2 Proof of Theorem 1.1

We note that the solution can be formally expressed as

$$\chi(t)=G(t)*_x \chi_0+H(t)*_x \chi_1+\int_0^t H(t-s)*_x g(\partial_x^2\chi,\partial_t\chi)ds.$$

Here G, H are the fundamental solutions of the corresponding linear equation, and the notation $*_x$ denotes the convolution with respect to x.

We recall several lemmas.

Lemma 1 (see [1]**).** Let $s\ge 0$, $1\le p\le 2$. Then the following estimates hold for $0\le k+l\le s$, $\varphi\in S$ (the class of Schwartz functions):

$$(1)\ \left\|\partial_x^{k+2}\mathrm{G}(t)*_x\varphi\right\|_{L^2}+\left\|\partial_t\partial_x^{k}\mathrm{G}(t)*_x\varphi\right\|_{L^2}\le C_{q,k}(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{2}+k\right)}\|\varphi\|_{L^p}+C_\mu(1+t)^{-\frac{\mu}{4}}\left\|\partial_x^{k+\mu+2}\varphi\right\|_{L^2},$$

$$(2)\ \left\|\partial_x^{k+2}\mathrm{H}(t)*_x\varphi\right\|_{L^2}+\left\|\partial_t\partial_x^{k}\mathrm{H}(t)*_x\varphi\right\|_{L^2}\le C_{q,k}(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{2}+k\right)}\|\varphi\|_{L^p}+C_\mu(1+t)^{-\frac{\mu}{4}}\left\|\partial_x^{k+\mu}\varphi\right\|_{L^2}.$$

By a little modification of the theorem 2.7 in [1], we have the following

Lemma 2 (see [1]**).** Let $s\ge\left[\frac{n+1}{2}\right]$ be an integer, $\tilde{\chi}(t):=G(t)*_x\chi_0+H(t)*_x\chi_1$ and $E_0:=\|\chi_0\|_{H^{s+2}}+\|\chi_1\|_{H^s}+\|\chi_0\|_{L^1}+\|\chi_1\|_{L^1}$. Suppose $\chi_0\in H^{s+2}\cap L^1$, $\chi_1\in H^s\cap L^1$. Then

$$\left\|\partial_x^{k+2}\tilde{\chi}(t)\right\|_{H^{s-\omega(k,n)}}+\left\|\partial_x^k\partial_t\tilde{\chi}(t)\right\|_{H^{s-\omega(k,n)}}\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\,\mathrm{E}_0.$$

Proof. Let $k, m\in Z^+$.

Let $p=1$ in Lemma 1, we have

$$\begin{aligned}\left\|\partial_x^{k+2}\tilde{\chi}(t)\right\|_{H^m}&\le\left\|\partial_x^{k+2}\mathrm{G}(t)*_x\chi_0(t)\right\|_{H^m}+\left\|\partial_x^{k+2}\mathrm{H}(t)*_x\chi_1(t)\right\|_{H^m}\\&\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\left\|\chi_0\right\|_{L^1}+C(1+t)^{-\frac{\mu_1}{4}}\left\|\partial_x^{k+\mu_1+2}\chi_0\right\|_{H^m}\\&\quad+C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\left\|\chi_1\right\|_{L^1}+C(1+t)^{-\frac{\mu_2}{4}}\left\|\partial_x^{k+\mu_2}\chi_1\right\|_{H^m}\\&\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}(\left\|\chi_0\right\|_{L^1}+\left\|\chi_1\right\|_{L^1})+C(1+t)^{-\frac{\mu_1}{4}}\left\|\chi_0\right\|_{H^{m+k+\mu_1+2}}\\&\quad+C(1+t)^{-\frac{\mu_2}{4}}\left\|\chi_1\right\|_{H^{m+k+\mu_2}}.\end{aligned}$$

Here $0\le k+m+\mu_i\le s,\quad (i=1,2)$.

Choose the smallest integers μ_i satisfying

$$\frac{\mu_i}{4}\ge\frac{n}{4}(k+\frac{1}{2}).$$

Then we have

$$\left\|\partial_x^{k+2}\tilde{\chi}(t)\right\|_{H^{s-\omega(k,n)}}\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\,\mathrm{E}_0.$$

Similarly, we have

$$\left\|\partial_x^k\partial_t\tilde{\chi}(t)\right\|_{H^{s-\omega(k,n)}}\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\,\mathrm{E}_0.$$

That is the conclusion.

Lemma 3 (see [2]**).** Let $1\le p,q,r\le\infty,\ \frac{1}{p}=\frac{1}{q}+\frac{1}{r}$ and $k\ge 0,\ \mathrm{m}\ge 1, n\ge 1$ be integers. Then

$$\left\|\partial_x^k\left(\xi^m\eta^n\right)\right\|_{L^p}\le C\left\|\xi\right\|_{L^\infty}^{m-1}\left\|\eta\right\|_{L^\infty}^{n-1}\left(\left\|\xi\right\|_{L^q}\left\|\partial_x^k\eta\right\|_{L^r}+\left\|\eta\right\|_{L^q}\left\|\partial_x^k\xi\right\|_{L^r}\right).$$

Just by a direct computation, we get

Proposition 1 (cf. [2]**).** Let $a\ge 0$ and $b\ge 0$ be real numbers. If $a+b\ge 1$, then there exists $C>0$ such that $\int_0^t(1+t-s)^{-a}(1+s)^{-b}ds\le C$.

Now we come to prove Theorem 1.1, we mimic the argument in [2], and define

$$\Upsilon := \left\{ \chi \in C^0(R^+; H^{s+2}) \bigcap C^1(R^+; H^s), \|\chi\|_Y < \infty \right\}$$

with

$$\|\chi\|_Y := \sup_{t\geq 0, 0\leq (k,n)\leq s} (1+t)^{\frac{n}{8}+\frac{k}{4}} (\|\partial_x^{k+2}\chi(t)\|_{H^{s-\omega(k,n)}} + \|\partial_x^k \partial_t \chi(t)\|_{H^{s-\omega(k,n)}}).$$

Denote

$$T[\chi](t) := G(t) *_x \chi_0 + \mathrm{H}(t) *_x \chi_1 + \int_0^t \mathrm{H}(t-\tau) *_x g(\Xi)(\tau) d\tau, \quad \text{with } \Xi := (\partial_x^2 \chi, \partial_t \chi).$$

By the assumption of f and Lemma 3, we have the following inequalities

$$\begin{aligned}\left\|\partial_x^k \left(g(\Xi) - g(\Upsilon)\right)(\tau)\right\|_{L^1} \leq C(\|\Xi(\tau)\|_{L^\infty} + \|\Upsilon(\tau)\|_{L^\infty})^{\alpha-2} \Big((\|\Xi(\tau)\|_{L^2} + \|\Upsilon(\tau)\|_{L^2}) \left\|\partial_x^k (\Xi - \Upsilon)(\tau)\right\|_{L^2} \\ + (\|\partial_x^k \Xi(\tau)\|_{L^2} + \|\partial_x^k \Upsilon(\tau)\|_{L^2}) \|(\Xi - \Upsilon)(\tau)\|_{L^2} \Big),\end{aligned}$$

and

$$\begin{aligned}\left\|\partial_x^k \left(g(\Xi) - g(\Upsilon)\right)(\tau)\right\|_{L^2} \leq C(\|\Xi(\tau)\|_{L^\infty} + \|\Upsilon(\tau)\|_{L^\infty})^{\alpha-2} \Big((\|\Xi(\tau)\|_{L^\infty} + \|\Upsilon(\tau)\|_{L^\infty}) \left\|\partial_x^k (\Xi - \Upsilon)(\tau)\right\|_{L^2} \\ + (\|\partial_x^k \Xi(\tau)\|_{L^2} + \|\partial_x^k \Upsilon(\tau)\|_{L^2}) \|(\Xi - \Upsilon)(\tau)\|_{L^\infty} \Big).\end{aligned}$$

Now we will prove that the mapping $\chi \to T[\chi]$ is contraction on $B_\varepsilon := \left\{ \chi \in Y; \|\chi\|_Y \leq \varepsilon \right\}$ for some $\varepsilon > 0$. This will be done in the following (S1)—(S4).

(S1). Set $s_n = \left[\frac{n}{2}\right] + 1$, $\theta_n = \frac{n}{2s_n}$. Taking $\chi \in Y$ and by Nirenberg's inequality, we have

$$\|\Xi(t)\|_{L^\infty} \leq C \|\Xi(t)\|_{L^2}^{1-\theta_n} \|\partial_x^{s_n} \Xi(t)\|_{L^2}^{\theta_n}.$$

(i) For $n = 1$, by assumption of s, we get $\|\Xi(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{8}} \|\chi\|_Y$ and

$$\|\partial_x^{s_n} \Xi(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{8}} \|\chi\|_Y. \text{ It yields } \|\Xi(t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{4}} \|\chi\|_Y.$$

(ii) For $n \geq 2$, since $s - \omega(0,n) \geq s_n$, we obtain $\|\Xi(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{8}} \|\chi\|_Y$ and $\|\partial_x^{s_n} \Xi(t)\|_{L^2} \leq C(1+t)^{-\frac{n}{8}} \|\chi\|_Y$. Then

$$\|\Xi(t)\|_{L^\infty} \leq C(1+t)^{-\frac{n}{8}} \|\chi\|_Y.$$

(S2). Take any $\chi, \eta \in Y$, and denote $\Xi := (\partial_x^2 \chi, \partial_t \chi)$, $\Upsilon := (\partial_x^2 \eta, \partial_t \eta)$. Then we have $T[\chi](t) - T[\eta](t) = \int_0^t \mathrm{H}(t-\tau) *_x \left(g(\Xi) - g(\Upsilon)\right)(\tau) d\tau$.

Assume $s \geq \omega(k,n)$, then we have

$$\begin{aligned}\left\|\partial_x^{k+2}(T[\chi](t)-T[\eta](t))\right\|_{H^m} &\le C\int_0^{\frac{t}{2}}\left\|\partial_x^{k+m+2}\mathrm{H}(t-\tau)*_x\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{H^0}d\tau\\ &+C\int_{\frac{t}{2}}^{t}\left\|\partial_x^{k+m+2}\mathrm{H}(t-\tau)*_x\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{H^0}d\tau\\ &=: I+II.\end{aligned}$$

Let $p=1$ in Lemma 1, we have

$$\begin{aligned}I &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{L^1}d\tau\\ &+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{\mu}{4}}\left\|\partial_x^{k+m+\mu}\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{L^2}d\tau.\end{aligned}$$

And

$$\begin{aligned}II &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{L^1}d\tau\\ &+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{\mu}{4}}\left\|\partial_x^{k+m+\mu}\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{L^2}d\tau.\end{aligned}$$

Then by a similar way as in [2], we can obtain

$$I+II\le C(1+t)^{-(\frac{n}{8}+\frac{k}{4})}(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y$$

with $0\le m\le s-\omega(k,n)$. That is,

$$\left\|\partial_x^{k+2}(T[\chi](t)-T[\eta](t))\right\|_{H^m}\le C(1+t)^{-(\frac{n}{8}+\frac{k}{4})}(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y$$

for $0\le m\le s-\omega(k,n)$.

So we have that

$$\sup_{t\ge 0}(1+t)^{\frac{n}{8}+\frac{k}{4}}\left\|\partial_x^{k+2}(T[\chi](t)-T[\eta](t))\right\|_{H^{s-\omega(k,n)}}\le C(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y.$$

(S3). In a similar way as in the part (S2), we can prove that

$$\sup_{t\ge 0}(1+t)^{\frac{n}{8}+\frac{k}{4}}\left\|\partial_x^{k}\partial_t(T[\chi](t)-T[\eta](t))\right\|_{H^{s-\omega(k,n)}}\le C(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y.$$

(S4). The estimates in (S2) and (S 3) imply that

$$\|T[\chi]-T[\eta]\|_Y\le C(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y.$$

So far we proved that $\|T[\chi]-T[\eta]\|_Y\le C_1\varepsilon^{\alpha-1}\|\chi-\eta\|_Y$, if $\chi,\eta\in B_\varepsilon$. By Lemma 2, we know that $\|G(t)*_x\chi_0+H(t)*_x\chi_1\|_Y\le C_2E_0$. So if E_0 and ε are sufficiently small, then we have

$$\|T[\chi]-T[\eta]\|_Y\le\frac{1}{2}\|\chi-\eta\|_Y.$$

It then yields that

$$\|T[\chi]\|_Y \le \|G(t) *_x \chi_0 + H(t) *_x \chi_1\|_Y + \frac{1}{2}\|\chi\|_Y \le \varepsilon .$$

Hence the mapping $\chi \to T[\chi]$ is contraction on B_ε. Then the fixed point theorem imply that there exists a unique fixed point $\chi \in B_\varepsilon$ satisfying $T[\chi] = \chi$. That is, this $\chi \in B_\varepsilon$ satisfies the equation

$$\chi(t) = G(t) *_x \chi_0 + H(t) *_x \chi_1 + \int_0^t H(t-s) *_x g(\partial_x^2 \chi, \partial_t \chi) ds.$$

So it is the solution to the semi-linear problem (P), and satisfies the corresponding decay estimates in Theorem 1.1.

Remark. In the proof above, we just sketched in some parts and left the details. The reader can refer to the paper [2] for similar argument.

3 Conclusion

We studied the Cauchy problem of a class of semi-linear plate type equation. We obtained the global existence (in time t) under the assumption of smallness of initial data, and some decay for solutions to this equation in terms of fixed point theorem. Our result is a generalization of the decay for the linear equation in [1].

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Liu Y, Ueda Y. Decay estimate and asymptotic profile for a plate equation with memory [J]. Journal of Differential Equations. 2020;268:2435–2463.

[2] Liu Y. Decay of solutions to an inertial model for a semilinear plate equation with memory [J]. Math. Anal. Appl. 2012;394:616-632.

[3] Da Luz CR, Charão RC. Asymptotic properties for a semi-linear plate equation in unbounded domains [J]. Hyper-bolic Differ. Equ. 2009;6:269–294.

[4] Sugitani Y, Kawashima S. Decay estimates of solutions to a semilinear dissipative plate equation [J]. Hyperbolic Differ. Equ. 2010;7:471–501.

[5] Liu Y. Kawashima S. Global existence and asymptotic behavior of solutions for quasi-linear dissipative plate equation. Discrete Contin. Dyn. Syst. 2011;29:1113-1139.

[6] Liu Y, Kawashima S. Global existence and decay of solutions for a quasi-linear dissipative plate equation. J. Hyperbolic Difffferential Equations. 2011;8:591-614.

[7] Liu Y, Kawashima S. Decay property for a plate equation with memory-type dissipation. Kinet. Relat. Mod. 2011;4:531-547.

[8] Liu Y. Asymptotic behavior of solutions to a nonlinear plate equation with memory. Commun. Pure Appl. Anal. 2017;16:533–556.

[9] Mao S, Liu Y. Decay of solutions to generalized plate type equations with memory. Kinet. Relat. Mod. 2014;7:121-131.

[10] Mao S, Liu Y. Decay property for solutions to plate type equations with variable coefficients. Kinet. Relat. Models. 2017;10:785–797.

[11] Liu Y, Kawashima S. Decay property for the Timoshenko system with memory-type dissipation, Math. Models Methods Appl. Sci. 2012;22:1–19.

[12] Liu Y, Kawashima S. Global existence and asymptotic decay of solutions to the nonlinear Timoshenko system with memory, Nonlinear Anal. TMA. 2013;84:1–17.

[13] Hosono T, Kawashima S. Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system, Math. Models Meth. Appl. Sci. 2006;16:1839-1859.

[14] Bradley ME, Lenhart S. Bilinear spatial control of the velocity term in a Kirchhoffff plate equation. Electronic J. Diffferential Equations. 2001;1-15.

[15] Buriol C. Energy decay rates for the Timoshenko system of thermoelastic plates. Nonlinear Analysis. 2006;64:92-108.

[16] Charão RC, Bisognin E, Bisognin V, Pazoto AF. Asymptotic behavior for a Dissipative plate equation in R^N with periodic coefficients, Electronic J. Differential Equations. 2008;1-23.

[17] Enomoto Y. On a thermoelastic plate equation in an exterior domain. Math. Meth. Appl. Sci. 2002;25:443-472.

[18] Sugitani Y, Kawashima S. Decay estimates of solutions to a semi-linear dissipative plate equation. J. Hyperbolic Differential Equations. 2010;7:471-501.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/79302