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On Almost Asymptotically Statistical Equivalent Sequences of Fuzzy Numbers

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Abstract

This paper presents the following definition which is a natural combination of the definitions for almost asymptotically equivalence and almost statistical convergence of fuzzy numbers. Let $\theta = (k_r)$ be a lacunary sequence. The two sequences X and Y of fuzzy numbers are said to be asymptotically $S_\theta^L(F)$ $-$statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon\right\}\right| = 0, \text{ uniformly in } m$$

(denoted by $X \overset{S_\theta^L(F)}{\sim} Y$) and simply almost asymptotically$-$statistical equivalent if $L = \bar{1}$. Also, we prove that some inclusion relations.

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1 INTRODUCTION

In 1965 L. A. Zadeh introduced the notion of fuzzy set theory. Since then it has been applied in almost all the branches of science and technology and has widely been investigated. Different types of fuzzy real numbers has been defined and applied for the studies in the recent years. It has been applied in sequence spaces and recently Tripathy and Borgohain (2008), Tripathy and Baruah (2010a, 2010b), Tripathy and Dutta (2007), Tripathy and Sarma (2008), Nanda (1989), Nuray and Savaş (1995), Savaş (2006, 2007, 2009a, 2009b), Savaş et al. (2010), Esi and Esi (2008) and many others have studied their different algebraic and topological properties.

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By a lacunarysequence $\theta = (k_r)$; $r = 0,1,2, \ldots$where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Let D denote the set of all closed and bounded intervals on ℝ, the real line. For $A, B \in D$, we define

$$d(A,B) = \max(|a_1 - b_1|, |a_2 - b_2|),$$

where$A = [a_1, a_2]$ and $B = [b_1, b_2]$. It is known that (D, d) is a complete metric space. A fuzzy real number X is a fuzzy set on ℝ, i.e. a mapping $X: \mathbb{R} \to I (= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $\mathbb{R}(I)$. Throughout the paper, by a fuzzy real number X, we mean that $X \in \mathbb{R}(I)$.

The $\alpha-$cut or $\alpha-$level set $[X]^\alpha$ of the fuzzy real number X, for $0 < \alpha \leq 1$, defined by $[X]^\alpha = \{t \in \mathbb{R}: X(t) \geq \alpha\}$; for $\alpha = 0$, it is the closure of the strong $0-cut$ i.e. closure of the set $\{t \in \mathbb{R}: X(t) > 0\}$. Throughout α means, $\alpha \in (0,1]$ unless otherwise it is stated.

The set of ℝ real numbers can be embedded in $\mathbb{R}(I)$ if we define $\overline{r} \in \mathbb{R}(I)$ by

$$\overline{r}(t) = \begin{cases} 1 & , \text{ if } t = r \\ 0 & , \text{ if } t \neq r \end{cases}.$$

The additive identity and multiplicative identity of $\mathbb{R}(I)$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. Then the arithmetic operations on $\mathbb{R}(I)$ are defined as follows:

$$(X \oplus Y)(t) = \sup\{X(s) \wedge Y(t-s)\}, t \in R,$$
$$(X \ominus Y)(t) = \sup\{X(s) \wedge Y(s-t)\}, t \in R,$$
$$(X \otimes Y)(t) = \sup\left\{X(s) \wedge Y\left(\frac{t}{s}\right)\right\}, t \in R,$$
$$(X/Y)(t) = \sup\{X(st) \wedge Y(s)\}, t \in R.$$

These operations can be defined in terms of $\alpha-$level sets as follows:

$$[X \oplus Y] = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha],$$
$$[X \ominus Y]^\alpha = [a_1^\alpha - b_1^\alpha, a_2^\alpha - b_2^\alpha],$$
$$[X \otimes Y]^\alpha = \left[\min_{i \in \{1,2\}} a_i^\alpha b_i^\alpha, \max_{i \in \{1,2\}} a_i^\alpha b_i^\alpha\right],$$
$$[X^{-1}]^\alpha = [(a_2^\alpha)^{-1}, (a_1^\alpha)^{-1}], a_i^\alpha > 0, i \in \{1,2\},$$

for each $0 < \alpha \leq 1$.

For r in ℝ and Xin $\mathbb{R}(I)$, the product rX is defined as follows:

$$rX(t) = \begin{cases} X(r^{-1}t) & , \text{ if } r \neq 0 \\ 0 & , \text{ if } r = 0 \end{cases}.$$

The absolute value $|X|$ of X in $\mathbb{R}(I)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\} & , \text{ if } t \geq 0 \\ 0 & , \text{ if } t < 0 \end{cases}$$

Kaleva and Seikkala (1984).

Let $\bar{d}: \mathbb{R}(I) \times \mathbb{R}(I) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \le \alpha \le 1} d([X]^\alpha, [Y]^\alpha).$$

Then $\bar{d}$ defines a metric on $\mathbb{R}(I)$. It is well known that $\mathbb{R}(I)$ is complete with respect to $\bar{d}$.

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real number X_0 if, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \ge n_0$.

A subset E of N is said to have density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

whereχ_E is the characteristic function of E.

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of natural numbers into $\mathbb{R}(I)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the $k-th$ term of the sequence.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to a fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that

$$\bar{d}(X_k, X_0) < \varepsilon$$

for $k \ge k_0$. Let $c(F)$ denote the set of all convergent sequences of fuzzy numbers.

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k: k \in N\}$ of fuzzy numbers is bounded. We denote by $\ell_\infty(F)$ the set of all bounded sequences of fuzzy numbers.It is straightforward to see that

$$c(F) \subset \ell_\infty(F)$$

and Nanda (1989) studied the spaces of bounded and convergent sequences of fuzzy numbers and showed that they are complete metric spaces.

Nuray and Savaş (1995) defined the notion of statistical convergence for fuzzy real number sequences and studied some properties. A fuzzy real number (X_k) is said to be statistically convergent to the fuzzy real number X_0, if for every $\varepsilon > 0$, $\delta\left(\left\{k \in N: \bar{d}(X_k, X_0) \ge \varepsilon\right\}\right) = 0$.

Savaş (2006) defined the notion of almost convergence for fuzzy real number sequences as follows: The sequence $X = (X_k)$ of fuzzy numbers is said to be almost convergent to a fuzzy number X_0 if

$$\lim_k \bar{d}(t_{km}(X), X_0) = 0, \text{uniformly in } m,$$

where

$$t_{km}(X) = \frac{1}{k+1} \sum_{i=0}^{k} X_{m+i}.$$

This means that for every $\varepsilon > 0$, there exists a $k_0 \in N$ such that

$$\bar{d}(t_{km}(X), X_0) < \varepsilon$$

whenever $k \geq k_0$ and for all m.

In Marouf (1993) Marouf presented definitions for asymptotically equivalent sequences of real numbers and asymptotic regular matrices. In Patterson (2003), Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. For sequences of fuzzy numbers, Savaş (2007) introduced and studied the concepts asymptotically equivalent and $\lambda -$statistical convergence. This notion is same as acceleration convergence recently studied by Tripathy and Mahanta (2010). The goal of this paper is to define almost asymptotically equivalent sequences, almost asymptotically statistical equivalent sequences for fuzzy numbers.

2 DEFINITIONS AND NOTATIONS

Definition 2.1. Two sequences X and Y of fuzzy numbers are said to be almost asymptotically equivalent if

$$\lim_k \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, \bar{1}\right) = 0, \text{uniformly in } m (\text{denoted by } X \overset{F}{\sim} Y).$$

Definition 2.2. Let $\theta = (k_r)$ be a lacunary sequence. A sequence of fuzzy numbers $X = (X_k)$ is said to be almost statistically convergent or $S_\theta -$*convergent* to the fuzzy number L if for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \bar{d}(t_{km}(X_k), L) \geq \varepsilon\right\}\right| = 0, \text{uniformly in } m$$

In this case we write $S_\theta - \lim X = L$or $X_k \to L(S_\theta)$.

The next definition is natural combination of Definitions 2.1. and 2.2.

Definition 2.3. Let $\theta = (k_r)$ be a lacunary sequence. Two sequences X and Y of fuzzy numbers are said to be almost asymptotically $S_\theta^L -$statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon\right\}\right| = 0, \text{uniformly in } m \ \ (\text{denoted by } X \overset{S_\theta^L(F)}{\sim} Y)$$

and simply almost asymptotically $S_\theta(F) -$statistical equivalent if $L = \bar{1}$.

Definition 2.4. Two sequences X and Y of fuzzy numbers are said to be almost asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}\left|\left\{k \leq n : \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon\right\}\right| = 0, \text{uniformly in } m (\text{denoted by } X \overset{S^L(F)}{\sim} Y)$$

and simply almost asymptotically statistical equivalent if $L = \bar{1}$.

Definition 2.5. Let $\theta = (k_r)$ be a lacunary sequence. Two sequences X and Y of fuzzy numbers are said to be strong $V_\theta^L(F)$ −asymptotically equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) = 0, \text{uniformly in } m (\text{denoted by } X \overset{V_\theta^L(F)}{\sim} Y)$$

and simply strong $V_\theta(F)$ −asymptotically statistical equivalent if $L = \bar{1}$.

In the special case $\theta = (2^r)$, we have the following definition:

Definition 2.6. Two sequences X and Y of fuzzy numbers are said to be strong Cesaro$C_1^L(F)$ −asymptotically equivalent of multiple L provided that

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) = 0, \text{uniformly in } m (\text{denoted by } X \overset{C_1^L(F)}{\sim} Y)$$

and simply strong Cesaro $C_1(F)$ −asymptotically equivalent if $L = \bar{1}$.

3 MAIN RESULTS

Theorem 3.1. Let $\theta = (k_r)$ be a lacunary sequence. The following conditions are satisfied:

(i) If $X \overset{V_\theta^L(F)}{\sim} Y$, then $X \overset{S_\theta^L(F)}{\sim} Y$.
(ii) If $X \in \ell_\infty(F)$and$X \overset{S_\theta^L(F)}{\sim} Y$, then $X \overset{V_\theta^L(F)}{\sim} Y$.
(iii) If$X, Y \in \ell_\infty(F)$ then $X \overset{V_\theta^L(F)}{\sim} Y$ if and only if $X \overset{S_\theta^L(F)}{\sim} Y$.

Proof. (i) Let $\varepsilon > 0$ and $X \overset{V_\theta^L(F)}{\sim} Y$, then

$$\sum_{k \in I_r} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \sum_{\substack{k \in I_r \\ \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon}} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon \left|\left\{k \in I_r : \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon\right\}\right|.$$

Therefore $X \overset{S_\theta^L(F)}{\sim} Y$.

(ii) Let X and Y are in $\ell_\infty(F)$ and $X \overset{S_\theta^L(F)}{\sim} Y$. Then we can assume that

$$\bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \leq T, \text{for all } k \text{and } m.$$

Given $\varepsilon > 0$

$$\frac{1}{h_r} \sum_{k \in I_r} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon}} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) < \varepsilon}} \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right)$$

$$\leq \frac{T}{h_r} \left|\left\{k \in I_r : \bar{d}\left(\frac{t_{km}(X_k)}{t_{km}(Y_k)}, L\right) \geq \varepsilon\right\}\right| + \varepsilon .$$

Therefore $X \overset{V_\theta^L(F)}{\sim} Y$.

(iii) Follows from (i) and (ii).

Theorem 3.2. Let $\theta = (k_r)$ be a lacunary sequence with $\lim\ \inf q_r > 1$,

$$X \overset{C_1^L(F)}{\sim} Y \text{ implies } X \overset{V_\theta^L(F)}{\sim} Y.$$

Proof. Suppose that $\lim\ \inf q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $X \overset{C_1^L(F)}{\sim} Y$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k \leq k_r : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{1}{k_r}\left|\left\{k \in I_r : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right|$$
$$\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right|,$$

this completes the proof.

Theorem 3.3. Let $\theta = (k_r)$ be a lacunary sequence with $\lim\ \sup q_r < \infty$,

$$X \overset{V_\theta^L(F)}{\sim} Y \text{ implies } X \overset{C_1^L(F)}{\sim} Y.$$

Proof. Suppose that $\lim\ \sup q_r < \infty$, then there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$. Let $X \overset{V_\theta^L(F)}{\sim} Y$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \geq R$

$$A_j = \frac{1}{h_j}\left|\left\{k \in I_j : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| < \varepsilon.$$

We can find $K > 0$ such that $A_j < K$ for all $j = 1,2,\ldots$. Now let n be any integer with $k_{r-1} < n < k_r$, where $r > R$. Then

$$\frac{1}{n}\left|\left\{k \leq n : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| \leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right|$$
$$= \frac{1}{k_{r-1}}\left|\left\{k \in I_1 : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{1}{k_{r-1}}\left|\left\{k \in I_2 : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| + \cdots$$
$$+ \frac{1}{k_{r-1}}\left|\left\{k \in I_r : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{k_1}{k_{r-1}k_1}\left|\left\{k \in I_1 : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}\left|\left\{k \in I_2 : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right|$$
$$+ \cdots + \frac{k_R - k_{R-1}}{k_{r-1}(k_R - k_{R-1})}\left|\left\{k \in I_R : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right| + \cdots$$
$$+ \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}\left|\left\{k \in I_r : \left|\frac{t_{km}(X_k)}{t_{km}(Y_k)} - L\right| \geq \varepsilon\right\}\right|$$

$$= \frac{k_1}{k_{r-1}k_1}A_1 + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}A_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}(k_R - k_{R-1})}A_R + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}A_r$$
$$\leq \left(\sup_{j\geq 1} A_j\right)\frac{k_R}{k_{r-1}} + \left(\sup_{j\geq R} A_j\right)\frac{k_r - k_R}{k_{r-1}} \leq K\,\frac{k_R}{k_{r-1}} + \varepsilon B.$$

This completes the proof.

Combining Theorem 3.1 and Theorem 3.2 we state the following without proof.

Corollary 3.4. Let $\theta = (k_r)$ be a lacunary sequence with $1 < lim\ inf q_r \leq \lim\ \sup q_r < \infty$, then

$$X \overset{V_\theta^L(F)}{\sim} Y \Leftrightarrow X \overset{C_1^L(F)}{\sim} Y.$$

4 CONCLUSION

The concepts of asymptotically equivalence and statistical convergence have been studied by various mathematicians. In this paper we introduce the concepts of asymptotically almost $\boldsymbol{S_\theta^L(F)}$ statistical equivalent and strong $\boldsymbol{V_\theta^L(F)}$ asymptotically equivalent sequences using by lacunary sequence. We also have given some relations between these sets.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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