Journal of Advances in Mathematics and Computer Science

Volume 39, Issue 3, Page 37-42, 2024; Article no.JAMCS.113269
ISSN: 2456-9968
(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Ultrafilter in Digraph: Directed Tangle and Directed Ultrafilter

Takaaki Fujita [a*]

[a] *Graduate School of Science and Technology, Gunma University, 1-5-1 Tenjin-Cho, Kiryu, Gunma 376-8515, Japan.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2024/v39i31874

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/113269

Short Research Article

Received: 13/12/2023
Accepted: 22/02/2024
Published: 29/02/2024

Abstract

Tangle is a concept in graph theory that has a dual relationship with tree-width which is well-known graph width parameter. Ultrafilter is a fundamental notion in mathematics. In this concise paper, we will reconsider the relationship between Tangle and Ultrafilter in digraph.

Keywords: Tangle, directed tangle; ultrafilter; directed ultrafilter; directed tree-decomposition; directed linear-branch-decomposition.

1 Introduction

Tangles, fundamental constructs in graph theory, are intricately linked with tree-width, a widely recognized metric for gauging the complexity of a graph. This connection has catalyzed significant scholarly interest, leading to a rich body of research as evidenced by references [1,2-11,12,13-19]. Similarly, ultrafilters have found utility across a diverse range of engineering disciplines, attracting attention from a broad spectrum of

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**Corresponding author: Email: t171d603@gunma-u.ac.jp;*

researchers, as indicated by references [1,20,21,22,23-32,33-35,36,37,38]. This widespread engagement underscores the perceived value of ultrafilter studies.

In this paper, we aim to reexamine the relationship between Tangles and Ultrafilters within the context of directed graphs.

2 Preliminaries

In this section, we present the fundamental definitions necessary for this paper. We consider a directed graph, often referred to as a digraph and denoted as G, in our context. We define an edge e as crossing from subgraph A to subgraph B if its tail resides in $V(A)$ but not in $V(B)$, and its head is situated in $V(B)$ but not in $V(A)$. A directed separation of a digraph G is represented as a pair (A, B) of subgraphs of G, satisfying the condition that their combined vertex set is $V(A) \cup V(B) = V(G)$. This separation can be characterized by the absence of cross edges, either from A to B or from B to A. The order of this separation is determined by the size of $V(A \cap B)$. Throughout this paper, we will use the variable k to represent a natural number.

2.1 Directed Tangle

The paper [39] provided "a proof of the Directed Tangle Tree-Decomposition Theorem". Furthermore, the paper [39] introduced "a method for constructing a directed tree-decomposition for any integer k, specifically designed to effectively distinguish all directed tangles with an order of k". And note that in recent times, there has been a growing interest in the graph width parameter in directed graphs [40-44]. Below, you will find the definition of a Directed Tangle.

Definition 1 [39]: Let G be a digraph. A set T of directed separations of order less than k in a digraph G is called a tangle of order k if: (DT1) For every directed separation (A, B) of G of order less than k, T contains either (A, B) or (B, A). (DT2) If $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in T$, then $V(A_1 \cup A_2 \cup A_3) \neq V(G)$.

2.2 Directed ultrafilter

First, we provide an explanation of Filters in Boolean Algebras. The definition of a filter in a Boolean algebra $(X,\cup,\cap)$ is given below.

Definition 2: In a Boolean algebra $(X,\cup,\cap)$, a set family $F \subseteq 2^X$ satisfying the following conditions is called a filter on the carrier set X.

(FB1) $A, B \in F \Rightarrow A \cap B \in F$,
(FB2) $A \in F, A \subseteq B \subseteq X \Rightarrow B \in F$,
(FB3) $\emptyset$ is not belong to F.

In a Boolean algebras $(X,\cup,\cap)$, A maximal filter is called an ultrafilter and satisfies the following axiom (FB4):
(FB4) $\forall A \subseteq X$, *either* $A \in F$ *or* $X / A \in F$.

In the context of directed graphs, we provide a definition for ultrafilters. The term "Directed Ultrafilter" is an extension of the definition of an Ultrafilter on a Boolean algebra to a directed graph. It's important to note that a Directed Ultrafilter is synonymous with a co-maximal Directed Ideal.

Definition 3: Let G be a digraph. A set F of directed separations of order less than k in a digraph G is called a Directed Ultrafilter of order k if:

(F1) For every directed separation (A, B) of G of order less than k, either (A, B) or (B, A) is an element of F.
(F2) If $(A_1, B_1) \in F$, $A_1 \subseteq A_2$, and (A_2, B_2) of G of order less than k, then $(A_2, B_2) \in F$.
(F3) If $(A_1, B_1) \in F$, $(A_2, B_2) \in F$, and $(A_1 \cap A_2, B_1 \cup B_2)$ of order less than k, then $(A_1 \cap A_2, B_1 \cup B_2) \in F$
(F4) For any directed separation (A, B) *such that* $V(A) = V(G)$, *we have* $(A, B) \in F$.

3 Cryptomorphism between Directed Tangle and Directed Ultrafilter

In this section, we demonstrate the cryptomorphism between Directed Tangle and Directed Ultrafilter. From this theorem, it becomes evident that the Directed Ultrafilter has a profound relationship with Directed tree-width.

Theorem 1. Let *G* be a digraph. *T* is a Directed Tangle of order *k* in a digraph *G* iff $F = \{(A,B) \mid (B,A) \in T\}$ is a Directed Tangle of order *k* in a digraph *G*.

Proof of Theorem 1:

To establish Theorem 1, we need to demonstrate that if T is a Directed Tangle of order *k*, then *F* is a Directed Ultrafilter of the same order, and conversely. We will rely on the definitions and conditions outlined in the preliminary section to construct this proof. Let us denote directed separations as *(A, B),* and define *F* as $F = \{(A,B) \mid (B,A) \in T\}$.

Conditions (F1) and (F4) are evidently satisfied.

To establish that F complies with (F2), we begin with an element (A_1, B_1) in F, indicating that (B_1, A_1) is in T. Now, let $A_1 \subseteq A_2$ and contemplate a separation (A_2, B_2) of order less than *k*. According to the definition of tangles, specifically axiom (DT1), T either contains (A_2, B_2) or (B_2, A_2). If T contains (A_2, B_2), it contradicts the assumption that (B_1, A_1) is in T, as this would violate (DT2). Therefore, T must contain (B_2, A_2), which implies that (A_2, B_2) is in F, thus satisfying condition (F2) for ultrafilters.

To ensure that *F* fulfills (F3), consider (A_1, B_1) and (A_2, B_2) in F, implying that (B_1, A_1) and (B_2, A_2) are in *T*. Now, contemplate a new separation $(A_1 \cap A_2, B_1 \cup B_2)$ of order less than *k*. Following a similar line of reasoning as for (F2), we conclude that this separation meets the conditions to be in *F,* thereby confirming that *F* satisfies condition (F3).

The reverse proof follows a similar approach in the opposite direction and is omitted for brevity.

In conclusion, we can establish that *T* is a Directed Tangle of order k if and only if $F = \{(A,B) \mid (B,A) \in T\}$ is a Directed Ultrafilter of order *k*, successfully substantiating Theorem 1. This proof is completed.

4 Future Tasks: Directed Linear-Branch-Decomposition

In this paper, we have explored the cryptomorphism between Directed Tangles and Directed Ultrafilters.

The reference [41] defines the directed branch-decomposition. Inspired by this, we propose the concept of a "Directed linear-branch-decomposition" and aim to explore its characteristics. For the notation and definitions used in the subsequent description, please refer to reference [41].

Definition 4. For any digraph *D*, let f_D be the function $f_D : 2^{E(D)} \to \mathbb{N}$ defined as $f_D(X)=$" $|S^V_X \cup S^V_{E(D)\setminus X}|$. A layout of *f(D)* on *E(D)* is called a directed branch decomposition of *D*. A tree is a caterpillar if its non-leaf vertices form a single path. The directed linear-branch-decomposition is a specialized form of directed branch decomposition where the underlying tree *T* is constrained to be a caterpillar. The directed linear branch width of *D* is defined as the width of the layout of *f(D)* on *E(D),* with the restriction that the decomposition tree is a caterpillar.

5 Conclusion

We intend to further investigate the properties of the aforementioned directed linear branch width. Moreover, we hypothesize that a concept, potentially the directed linear tangle, may exhibit a duality with the directed linear branch width. Our research in this area will be ongoing.

Furthermore, a range of graph width parameters has been established, as indicated in references [34,45,46]. We aim to explore the properties that manifest when these parameters are extended to directed graphs.

Disclaimer

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https://www.researchgate.net/publication/374112335_Ultrafilter_in_digraph_Directed_Tangle_and_Directed_Ultrafilter
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Acknowledgements

I would like to express my sincere gratitude to all those who have supported me in completing this paper.

Competing Interests

Author has declared that no competing interests exist.

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