A Scientific Computing Analysis of Financial Black-Scholes and Monte Carlo Differential Equation: An American Option

Samson Oluyomi Akintola [a*]

[a] *Department of Mathematics, University of Ibadan, Ibadan, Nigeria.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Original Research Article

ABSTRACT

This study presents a systematic computing analysis of financial models, precisely focusing on the Black-Scholes and Monte Carlo derivative equations, to evaluate American options. American selections are exercised at any time before expiration, posing unique challenges in financial modelling due to their complex early exercise features. The Black-Scholes formulation gives a foundational framework for choice pricing, utilizing partial derivative formulations to estimate the fair value of options under definite assumptions. Nevertheless, because of its restriction, Monte Carlo computations are taken to give a better simulation scheme to overcome the posed challenges by computing wider likely underlying price path assets. This study implements a computationa approach to compare the efficacy of the Black-Scholes formulation and Monte Carlo methods in selected American pricing. A numerical scheme for solving the Black-Scholes derivative systems and a variance reduction technique for enhancing the effectiveness of Monte Carlo simulations are adopted. Our analysisl

*Corresponding author: E-mail: akintolayomi@yahoo.com;

reveals that while the Black-Scholes model provides a useful approximation, Monte Carlo simulations deliver more accurate and flexible results for American options, especially in scenarios with substantial volatility and early exercise potential. The outcomes underscore the importance of sophisticated numerical methods in financial engineering and highlight the trade-offs between analytical tractability and numerical precision.

1 INTRODUCTION

Finance is the fastest-rising and highly speedily varying area in corporate organisation. Following its swift variation, financial contemporary apparatuses are now really intricate. As such, innovative scientific formulations are indispensable for implementing and pricing these novel monetary apparatuses. The finance commercial world, earlier controlled by corporate apprentices, is nowadays governed by computer and mathematician experts. In the early 1970s, Fisher Black, Robert Merton, and Myron Scholes accomplished essential success in the financial complex pricing tools by formulating a well-known model called the Black-Scholes technique. In 1997, the strength and essential of the modelled technique was acknowledged globally when Robert Merton and Myron Scholes were given an Economics Nobel Prize. The Black-Scholes formulation demonstrated the importance of mathematics in the finance field. This led to the evolution and victory recorded in financial engineering or financial mathematical fields.

The cell (put) option owner has no obligation but the right to buy (sell) a primary benefit at the price exercise. The European option may be practised or executed at the expiration time only, but the American option can be any time executed and practised until the expiration time. A closed-form solution for the European option is developed from the ideas in [1,2]. For the American option, due to the early possibility practice, the problem of pricing gives a complicated analytical calculation. The scholars [3,4] presented that the American valuation option establishes a boundary-free problem for time changes in the boundary to maturity, frequently referred to as boundary optimum exercise [5,6]. Therefore, finance scientists have considered approaches to correctly and rapidly find the boundary optimum exercise. The techniques are fundamentally different, i.e., approximating analytical schemes as demonstrated by [7-9] and computational techniques as

illustrated by [10,11]. Wu and Kwok [12] find an explicit and precise solution for the Black-Scholes model in evaluating the Put American option through the Taylor series of indefinitely several parameters. Their study gives an exceptional outcome for the Put American option valuations; meanwhile, it is complex to carry out the numerical analysis and solution. The infinite totality yields several computational errors.

Michael et a. [13] studied expanded the Wu and Kwok [12] to American pricing option with wide-ranging distribution procedures. The most common computational scheme for the American pricing option includes the finite difference technique by Brennan and Schwartz [14], binomial scheme by Cox et al. [15], Monte Carlo computing scheme by Glasserman [16], least squares technique by Tilley [17], integral-equation technique by Brandimarte [18], and the Laplace transform procedure by Boyle et al. [19] are recursive time-depended. The lifetime discretization of options is the basis of their idea in calculating the backward boundary optimal time exercise. Since recursive time-dependent ways produced a repeated time step calculation, it requires quick time computations and lower pricing errors. Moreover, the front-fixing schemes [12, 20] formulated use a quasilinear transformation to solve the resultant quasilinear model and fix the boundary. A Secant technique formulated by Wilmott et al. [21] is required to provide a solution to a quasilinear equation, and a boundary-moving procedure presented by Geske and Johnson [22] changes the resultant boundary-free linear partial derivative model into a fixed-boundary linear sequence PDE model. Recently, Han and Wu [23] developed a novel corrector-predictor method for the Put American pricing option by applying the Black-Scholes formulation. After that, Wilmott et al. [21] presented an expansion of Han and Wu [23] to provide a valuation for the Put American option with volatility stochastic formulation. The investigation adopts a numerical procedure to analyse the effectiveness of the Black-Scholes formulation and

Monte Carlo schemes for American pricing options. Numerical procedures for solving the Black-Scholes partial derivative models and variance reduction techniques for enhancing the proficiency of Monte Carlo simulations are adopted.

2 THE BLACK-SCHOLES FORMULATION

Following the derivative of lognormal dynamics, the Merton, Myron Scholes, and Fischer Black assumptions are developed for the European pricing option formulation. Additional assumptions were made as:

- The stock rate of return probability is lognormally distributed with the same return free-risk and mean,

- Transaction taxes or costs ignored,

- There is no arbitrage risk-free occur,

- During the option life dividends do not exist,

- The interest risk-free rate in time is constant and known,

- The return variance life option is fixed,

- There is continuous trading of asset with continuous varying price.

Given the stock price as

$$dS = \mu S dt + \sigma S dW \tag{1}$$

where W is the Wiener process, σ dentes volatility, and μ defines trend. The call option is f and other contingent derivative presents S where f is a function t and S. Thus, by **Ito's lemma**

$$df = \left[\frac{\partial F}{\partial S} \mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 \right] dt + \frac{\partial F}{\partial S} \sigma S dW \tag{2}$$

Equations (1) and (2) in discrete for

$$\partial S = \mu S \partial t + \sigma S \partial W$$
$$\partial f = \left[\frac{\partial F}{\partial S} \mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 \right] \partial t + \frac{\partial F}{\partial S} \sigma S dW \tag{3}$$

The underlying Wiener procedure for S and f are equal and can be removed through right choice of portfolio for the derivative and stock. A portfolio is chosen as:

$$+ \frac{\partial f}{\partial S} : \text{ shares and } -1 : \text{derivative}$$

It is a derivative of short holder and share long amount of $\partial f / \partial S$. Define Θ to be the portfolio value to have

$$\Theta = \frac{\partial f}{\partial S} S - f. \tag{4}$$

The portfolio value is the change in $\partial \Theta$ with time variance of ∂t is describes as:

$$\partial \Theta = \frac{\partial f}{\partial S} \delta S - \partial f. \tag{5}$$

Using model (3) in model (5) gives

$$\partial \Theta = \left[-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right] \partial t \tag{6}$$

By the removal of the term ∂W, the portfolio less-risk. A return must then earn equivalent to other free-risk securities. Hence,

$$\delta\Theta = r\Theta\partial t \tag{7}$$

where the interest free-risk rate is r. Applying (4) and (6) on (7) results to

$$\left[\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial t}\right]\partial t = -r\left[\frac{\partial f}{\partial s}S - f\right]\partial t. \tag{8}$$

To have,

$$\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + rS\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} = fr \tag{9}$$

This gives Black-Scholes-Merton derivative model.

Solving model (9) results to an European pricing option analytical, which is exercised at the due time and not applicable for early exercise pricing. Therefore, the European boundary constraints option can be used on model (9).

2.1 The Black-Scholes Model Solution

The condition for payoff is given as $\max(S - N, 0) = f(S, t = T)$. The boundary upper and upper constraints are consider as $c(S, K, t), C(S, K, t) \geq -Ke^{(t-T)} + S_t$ and $c(S, t, K), C(S, t, K) \leq S_t$. These are the satisfied PDE constraints

Let $\tau = T - t$, where T denotes time expiration and t is time; thus, mode; (9) becomes

$$\frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial S}rS - fr \tag{10}$$

Let

$$\frac{\partial f}{\partial S} = \frac{1}{S}\frac{\partial f}{\partial y}$$

and

$$\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}\frac{\partial f}{\partial y} + \frac{1}{S^2}\frac{\partial^2 f}{dy^2} \tag{11}$$

If $y = \ln S,$, new notation is introduced

$$w(y, \tau) = e^{r\tau}f(y, \tau)$$

Using (11), the diffusion PDE Black-Scholes model gives

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} + \left[r - \frac{\sigma^2}{2} \right] \frac{\partial w}{\partial y} \tag{12}$$

with a basic normal solution as:

$$\phi(y, \tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left[-\frac{[(r - \frac{\sigma^2}{2})\tau]^2 + y}{2\sigma^2\tau} \right] \tag{13}$$

Hence, equation (13) results to:

$$w(y, z) = \int_{-\infty}^{\infty} \phi(y - \xi, \tau) w(\xi, 0) d\xi \tag{14}$$

Introducing the payoff constraint to have the basic solution for equation (13) as:

$$w(y, \tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} \max(e^\xi - K, 0) exp\left[\frac{[y - \xi + (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau} \right] d\xi$$

$$= \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{\ln K}^{\infty} (e^\xi - K, 0) \exp\left[-\frac{[(r - \frac{\sigma^2}{2})\tau - \xi + y]^2}{2\sigma^2\tau} \right] d\xi \tag{15}$$

The normal variable distribution function is presented as

$$K(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{16}$$

We can express (15) as

$$w(y, \tau) = \frac{1}{\sqrt{2\sigma^2\tau\pi}} \int_{\ln K}^{\infty} \exp\left[-\frac{(A - \xi)^2}{2\tau\sigma^2} \right] e^\xi d\xi$$

$$- \frac{N}{\sqrt{2\sigma^2\tau\pi}} \int_{\ln K}^{\infty} \exp\left[-\frac{(A - \xi)^2}{2\tau\sigma^2} \right] d\xi \tag{17}$$

where $A = y + (r - \frac{\sigma^2}{2})\tau = \ln S + (r - \frac{\sigma^2}{2})\tau$. The RHS second term of (17) can be simplified to have

$$z = \frac{(A - \xi)}{\sqrt{\sigma^2\tau}} \tag{18}$$

then $d\xi$ gives

$$d\xi = -\sqrt{\sigma^2 \tau} dz$$

The limit $\xi = \infty$ and $z = -\infty$, are taken for (17) using (18),

$$z = \frac{A - \ln K}{\sqrt{\sigma^2 \tau}} = \frac{(r - \frac{\sigma^2}{2})\tau + \ln S - \ln K}{\sqrt{\sigma^2 \tau}} \equiv d_2 \quad \text{where} \quad \xi = \ln K \tag{19}$$

Carry out variable from ξ to z changes, the equation (17) second term gives

$$\frac{K}{\sqrt{2\pi}} \int_{d_2}^{d_1} e^{-z^2/2} dz = -\frac{K}{\sqrt{2\pi x}} \int_{\infty}^{1} e^{-z^2/2} dz$$

The equation (17) first term integral results to

$$\exp\left[-\frac{(A - \xi)^2}{2\tau\sigma^2}\right] e^{\xi}$$

$$= \exp\left[-\frac{A^2 - 2(\tau\sigma^2 + A)\xi + \xi^2 \xi^2}{2\tau\sigma^2}\right]$$

$$= \exp\left[-\frac{A^2 - (\tau\sigma^2 + A) + (\tau\sigma^2 + A)^2 - 2(\tau\sigma^2 + A)\xi + \xi^2}{2\tau\sigma^2}\right]$$

$$= \exp\left[-\frac{[-(\tau\sigma^2 + A) + \xi]^2}{2\tau\sigma^2} + A + \frac{1}{2}\tau\sigma^2\right]$$

$$= e^{A + \frac{1}{2}\tau\sigma^2} \exp\left[-\frac{[-(\tau\sigma^2 + A)]^2 + \xi}{2\tau\sigma^2}\right] \tag{20}$$

By the A description,

$$e^{A + \frac{1}{2}\tau\sigma^2} = e^{\tau r + y} = Se^{\tau r} \tag{21}$$

Use (20, 21) on the equation (17) first term gives

$$\frac{1}{\sqrt{2\sigma^2 \tau\pi}} Se^{\tau r} \int_{\ln K}^{\infty} \exp\left[-\frac{[-(\tau\sigma^2 + A) + \xi]^2}{2\tau\sigma^2}\right] d\xi. \tag{22}$$

By variable changes,

$$\frac{1}{\sqrt{2\pi}} Se^{r\tau} \int_{-\infty}^{d_1} e^{-\frac{z^2}{2}} dz = Se^{r\tau} K(d_1) \tag{23}$$

The last line of (15) can be written as;

$$w(y,\tau) = e^{r\tau}SN\Big[\frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big] - KN\Big[\frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}\Big] \tag{24}$$

this implies that

$$C = -KNe^{-\tau r}(d_2) + NS(d_1) \tag{25}$$

where

$$d_1 = \frac{\tau(\frac{\sigma^2}{2} + r) + \ln(\frac{S}{K})}{\tau\sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sqrt{\sigma^2\tau}. \tag{26}$$

Thus, the time zero pricing for Black-Scholes model of a paying stock non-dividend European call option .
By using $P = -S + Ke^{-Tr} + c.$, The analytical put European model is obtained as

$$P = -NS(-d_1) + KNe^{-\tau r}N(-d_2). \tag{27}$$

The analytical put and call European model have become very popular in the financial world since it is easily used to determine European options.

3 The American Option

For the call American option, the holder has the choice and not responsibility to buy a prescribed asset price from the start time to the expiry time; hence, the American option has higher values potentially.

Given S in between $P(S,t) < \max(E - S, 0)$ for an option exercise, there exist a arbitrage privilege with free-risk profit of $E - S - P$. Therefore, it can be concluded that constraints must be imposed when permitting early exercise.

$$V(S,t) \geq \max(S - E, 0)$$

The American option-valuation is complex than its counterpart European option-valuation since the option value only needs to be determined. A unique specification of American option-valuation is done by the conditions

- the value for the option is equal or higher than the function of the payoff;
- an inequality is used to replace Black-Scholes model;
- a S continuous function for option value exist and;
- the delta option ($\frac{\partial V}{\partial S}$) is continuous.

Holding a call American option to maturity is optimal since no payment of dividend on the underlying asset.

3.1 The Put American Optimal Price Exercise

Let the Put American option be denoted by $V(S,t)$, asset price is given as S and current time is t. Black and Scholes [2] gives the put and call value option V in partial derivative model as:

187

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial S}rS - Vr = 0, \tag{28}$$

where interest free-risk rate is r and underlying volatility asset is defined as S. Eq. (28) is described Black-Scholes model; thus, σ and r and constant. Equation (28) is the Black-Scholes financial derivative. For put American and European options, there exist far-field boundary constraint

$$\lim_{S \to \infty} V(S,t) = 0 \tag{29}$$

this implies put option is worthless as the underlying price asset is getting increase unlike European option. A asset critical price $S_r(t)$ is given, which is equal or below its optimal exercise of the put American option. With optimal price exercise, according to Wilmott et al. [21], the boundary constraint of the optimal price exercise defined as $S = S_r(t)$ is given as

$$\begin{cases} V(S_r(t), t) = -S_r(t) - X \\ \frac{\partial V}{\partial t}(S_r(t), t) + 1 = 0, \quad \text{or a continuous } \frac{\partial V}{\partial t} \text{ on } S = S_r(t), \end{cases} \tag{30}$$

for which option strice price is X. From mathematical perspective, free value boundary model is constituted, where the location of the boundary formed the solution part of the model. Though, in terms of V, the governing derivative model is linear but made nonlinear due to the unknown boundary constraint. The unknown product function $\frac{1}{S_r}\frac{\partial v}{\partial s}\frac{dS_f}{dt}$, which exist in the partial derivative model gives a better measure for nonlinearity strength. The put value option been equal to the payoff setup functions ends the condition

$$V(S, T) = \max\{X - S, 0\}, \tag{31}$$

where the option expiration time is T. As such, equations (28)-(31) give a derivative system, which the solution resulted in an American option value before expiration of time T at any price S.

To appropriately solve the system, non-dimensionl variables are introduced to have a dimensionless system

$$S' = \frac{S}{X}, \qquad V' = \frac{V}{X}, \qquad \tau' = \frac{\sigma^2}{2}(T - t) = \frac{\sigma^2}{2}.\tau.$$

The dimensionless system with prime dropped becomes

$$\begin{cases} -V\gamma - \frac{\partial V}{\partial \tau} + S\gamma\frac{\partial V}{\partial S} + S^2\frac{\partial^2 V}{\partial S^2} = 0, \\ \frac{\partial V}{\partial S}(S_r(\tau), \tau) = -1, \\ V(S, 0) = \max\{1 - S, 0\}, \\ \lim_{S \to \infty} V(S, \tau) = 0, \\ V(S_r(\tau), \tau) = -S_r(\tau) + 1, \end{cases} \tag{32}$$

where relative interest rate to the volatility price asset is $\gamma \equiv \frac{2r}{\sigma^2}$. The dimensionless derivative system (32) depicts a family of two-parameter solution. The solutions are the relative interest rate, γ and total dimensionless time,

$\tau_{exp} = \frac{\sigma^2}{2}.T$, for option initial time $t = 0$ to the expire time T. Due to the introduced expire time γ as the difference between times T and t, an terminal constraint (31) gives an initial constraint (32). Consider a new function $U(S, \tau)$ described as

$$U = \begin{cases} S - 1 + V, & \text{if } S_r \leq S < 1, \\ V, & \text{if } S \geq 1, \end{cases} \tag{33}$$

The derivative model (32) can be presented in two equations and boundary constraints

$$\begin{cases} S^2 \frac{\partial^2 U}{\partial S^2} + \gamma S \frac{\partial U}{\partial S} - \frac{\partial U}{\partial \tau} - U\gamma = \gamma, \\ U(S_f(\tau), \tau) = 0, & \text{if } S_f \leq S < 1, \\ \frac{\partial U}{\partial S}(S_f(\tau), \tau) = 0, \\ U(S, 0) = 0, \end{cases} \tag{34}$$

$$\begin{cases} S^2 \frac{\partial^2 U}{\partial S^2} + \gamma S \frac{\partial U}{\partial S} - \frac{\partial U}{\partial \tau} - U\gamma = 0, \\ \lim_{S \to \infty} U(S, \tau) = 0, & \text{if } S \geq 1, \\ U(S, 0) = 0, \end{cases} \tag{35}$$

As presented, the constraints in (34) and (35) now appeared easier to handle than that of (31). The moving boundary constraints $S = S_r(t)$ is homogeneous because the derivative model of (34) is non-homogeneous, which facilitates considerable solution step. To establish V as a contentious function $V(S, \tau)$, its derivatives are needed on the boundary $S = 1$, which resulted into matching interfacial constraints

$$\lim_{S \to 1^+} U = \lim_{S \to 1^-} U, \tag{36}$$

$$1 + \lim_{S \to 1^+} \frac{\partial U}{\partial S} = \lim_{S \to 1^-} \frac{\partial U}{\partial S}, \tag{37}$$

where 1^+ and 1^- respectively indicate S is approaching 1 from the right and left. Taking Laplace transform of the systems (34)-(37) for the optimal price exercise $S_r(t)$ and option price $U(S, \tau)$, the existence conditions for Laplace transform are satisfied as given by Karatzas [24]. All quantities shall be denoted in the Laplace space as:

$$\mathcal{L}U(S, \tau) = \int_0^\infty e^{-p\tau} U(S, \tau) d\tau = U(S, p), \qquad \mathcal{L}S_r(\tau) = \int_0^\infty e^{-p\tau} S_r(\tau) d\tau = \bar{S}_f(p).$$

The Laplace transform of the systems (34)-(37) give the respective ordinary derivative equations in terms of p with the substitution of the initial constraints

$$\begin{cases} -[p\bar{U} - 0] + S^2 \frac{d^2\bar{U}}{dS^2} + \gamma S \frac{d\bar{U}}{dS} - \gamma \bar{U} = \frac{\gamma}{p}, \\ \bar{U}(pS_f, p) = 0, \\ \frac{d\bar{U}}{dS}(p\bar{S}_f, p) = 0, \end{cases} \tag{38}$$

$$\begin{cases} -[p\bar{U} - 0] + S^2 \frac{d^2\bar{U}}{dS^2} + \gamma S \frac{d\bar{U}}{dS} - \gamma \bar{U} = 0, \\ \lim_{S \to \infty} \bar{U}(S, p) = 0, \end{cases} \tag{39}$$

$$\begin{cases} \bar{U}(1^+, p) = \bar{U}(1^-, p), \\ \frac{1}{p} + \frac{d\bar{U}}{dS}(1^+, p) = \frac{d\bar{U}}{dS}(1^-, p) \end{cases} \tag{40}$$

It is observed that under the Laplace transform derivation model of (38) and (39), the matching interfacial constraints in (40) and the boundary far-field constraint of (39) are direct. Meanwhile, a Laplace transformation is carried out on the boundary constraints of (34), which is demonstrated on the moving boundary $S_r(\tau), S$ in

$$\mathcal{L}U(S, \tau) = \int_0^\infty U(S, \tau) e^{-\tau p} d\tau,$$

According to an approximate steady pseudo-state, if a slow movement optimal boundary exercise assumed and compared to option price diffusion, S will remain constant in the Laplace transform and be replaced interfacial

constraint $S = S_r(\tau)$ (i.e., $\mathcal{L}S_r(\tau) = \mathcal{L}S \Rightarrow \frac{S}{p} = \bar{S}_r$). Also, the same argument is established for the second moving boundary constraint of (38). From the Stefan classical model, the steady pseudo-state approximate solution gives indefinite speed at $t = 0$, like the Stefan model exact solution (see Kemna and Vorst [25]). The system derivative solutions of (38)-(40) is done as:

$$\bar{U} = \begin{cases} D_2 S^{q2} + D_1 S^{q1} - \frac{\gamma}{p(p+\gamma)}, & \text{if } S_f \leq S < 1, \\ D_3 S^{q1} + D_4 S^{q2}, & \text{if } S \geq 1, \end{cases} \tag{41}$$

where the characteristic roots are given as q_1 and q_2 respectively for the corresponding homogeneous equation

$$q_{1,2} = \frac{1-\gamma}{2} \pm \sqrt{\left(\frac{1-\gamma}{2}\right)^2 + (p+\gamma)}, \tag{42}$$

and the unknown complex arbitrary constants are presented as D_1, D_2, D_3 and D_4, which are to be evaluated for all conditions to be satisfied. Hence, equation (42) can be expressed as

$$q_{1,2} = m \pm \sqrt{m^2 + (\gamma + p)} = m \pm \sqrt{a^{2+p}}, \tag{43}$$

where $m = -\frac{(\gamma-1)}{2}$ and $a = \frac{\gamma+1}{2}$. If γ varied in $(0, \infty)$, a is positive while m is either negative or positive. In fact, m takes value between $(\frac{1}{2}, -\infty)$. Thus, m and a are associated as

$$a + m = 1, \qquad a - m = \gamma, \qquad a^2 = \gamma + m^2.$$

Taking an appropriate contour for the Laplace inverse transform, the q_1 is the positive real part and the q_2 is the negative real part. As such, for the boundary far-field constraint (29) to be satisfied, D_3 must be of zeroth order. The satisfying other boundary constraints and the interface constraints of (38)-(40) results in algebraic equations

$$\begin{cases} D_1(p\bar{S}_f)^{q1} + D_2(p\bar{S}_f)^{q2} = \frac{\gamma}{p(p+\gamma)}, \\ D_1 q_1(p\bar{S}_f)^{q1-1} + D_2 q_2(p\bar{S}_f)^{q2-1} = 0, \\ D_1(1)^{q1} + D_2(1)^{q2} - \frac{\gamma}{p(p+\gamma)} = D_4(1)^{q2}, \\ D_1 q_1(1)^{q1} + D_2 q_2(1)^{q2} = D_4 q_2(1)^{q2} + \frac{1}{p}, \end{cases} \tag{44}$$

the solution yields a elegant and simple optimal price exercise formula in the Laplace space

$$\bar{S}_f = \frac{1}{p}\left[\frac{\gamma q_2}{\gamma q_2 - (p+\gamma)}\right]^{\frac{1}{q_1}}, \tag{45}$$

therefore, three unknown constants from $U(S, \tau)$ is to be evaluated for the option price. The optimal price exercise $S_f(\tau)$ is obtained from the inversion of (45). Hence, the Laplace inverse transform gives

$$\bar{S}_f(\tau) = \frac{1}{2\pi i}\int_{\mu-\infty i}^{\mu+\infty i} \frac{e^{p\tau}}{p} \cdot \left[\frac{(p+\gamma)}{\gamma q_2 - (p+\gamma)}\right]^{\frac{1}{q_1}} dp,$$

$$= \frac{1}{2\pi i}\int_{\mu-\infty i}^{\mu+\infty i} \frac{e^{p\tau}}{p} \cdot \exp\left\{\frac{-\log\left[1 - \frac{(p+\gamma)}{\gamma(b-\sqrt{p+a^2})}\right]}{b + \sqrt{p+a^2}}\right\} dp, \tag{46}$$

to ensure that $Re(q_2) < 0$ and $Re(q_1) > 0$, it is sufficient to assume any μ, such that

$$\mu > 0 \tag{47}$$

Following Cauchy's residue theorem defined as

$$\sum_{j=1}^{6} \int_{C_j} e^{p\tau} \bar{S}_f(p) dp = 2\pi i \sum_{k=1}^{n} Res_{p=p_k} \{e^{p\tau}\} \bar{S}_f(p)$$

$$= 2\pi i \sum_{k=1}^{n} Res_{p=p_k} \left\{ \frac{e^{p\tau}}{p} \exp \left\{ \frac{-\log\left[1 - \frac{(p+\gamma)}{\gamma(b-\sqrt{p+a^2})}\right]}{b+\sqrt{p+a^2}} \right\} \right\}, \tag{48}$$

One simple isolated pole $e^{p\tau}$ exist when $p = 0$. As given by Broadie and Glasserman [26], an optimal perpetual price exercise residue $e^{p\tau}$ is determined. In relation to the derivation, the integrand residue of (46) is obtained as:

$$I_3 + I_5 = 2ie^{-a^2 r} \int_0^\infty \frac{e^{-\rho\tau}}{\rho + a^2} \cdot Im \left\{ \exp\left[\frac{-\log\left(1 + \frac{(b-i\sqrt{\rho})}{\gamma}\right)}{b - i\sqrt{\rho}} \right] \right\} d\rho, \tag{49}$$

In (49), $Im\{\cdot\}$ denotes complex function imaginary part. On dividing equation (49) on both sides by 2π and use the outcome on equations (46) and (48), an analytical optimal price exercise formula is gotten S_r

$$S_f(\tau) = \frac{\gamma}{(1+\gamma)} + \frac{e^{-a^2\tau}}{\rho} \int_0^\infty \frac{e^{-\rho\tau}}{a^2 + \rho} e^{-f_1(\rho)} \sin[f_2(\rho)] d\rho, \tag{50}$$

where

$$f_1(\rho) = \frac{1}{(\rho + b^2)} \left[b \ln\left(\frac{\sqrt{\rho + a^2}}{\gamma}\right) + \sqrt{\rho} \tan^{-1}\left(\frac{\sqrt{\rho}}{a}\right) \right],$$

$$f_2(\rho) = \frac{1}{(b^2 + \rho)} \left[\sqrt{\rho} \ln\left(\frac{\sqrt{a^2 + \rho}}{\gamma}\right) - b \tan^{-1}\left(\frac{\sqrt{\rho}}{a}\right) \right],$$

3.2   The Put American Price Option

The respective D_1, D_2 and D_4 can be determined once $\bar{S}_r(p)$ is known from (17) and expressed in Laplace term p as

$$D_1 = \frac{1}{(S_r p)^{q1}} \cdot \frac{q_2}{(q_2 - q_1)} \cdot \frac{\gamma}{(\gamma + p)p}$$
$$= \left[1 - \frac{\gamma + p}{(m - \sqrt{a^2 + p})\gamma}\right] \cdot \frac{m - \sqrt{a^2 + p}}{2\sqrt{a^2 + p}} \cdot - \frac{\gamma}{(\gamma + p)p}. \tag{51}$$

$$D_2 = \frac{1}{(S_r p)^{q2}} \cdot \frac{q_1}{(q_1 - q_2)} \cdot \frac{\gamma}{(\gamma + p)p}$$
$$= \left[1 - \frac{\gamma + p}{(m - \sqrt{a^2 + p})\gamma}\right]^{\frac{q_2}{q_1}} \cdot \frac{\sqrt{a^2 + p} + m}{2\sqrt{a^2 + o}} \cdot \frac{\gamma}{(\gamma + p)p}. \tag{52}$$

$$D_4 = D_2 + D_1 - \frac{\gamma}{(\gamma + p)p},$$
$$= \frac{\gamma}{(\gamma + p)p} \cdot \left\{ -\left[1 - \frac{\gamma + p}{(m - \sqrt{a^2 + p})\gamma}\right] \cdot \frac{m - \sqrt{p + a^2}}{2\sqrt{p + a^2}} + \left[1 - \frac{\gamma + p}{(b - \sqrt{a^2 + p})\gamma}\right]^{\frac{q_2}{q_1}} \cdot \frac{\sqrt{a^2 + p} + m}{2\sqrt{a^2 + p}} - 1 \right\}. \tag{53}$$

As such, $U(S, \tau)$ is expressed as

$$U(S, \tau) = \frac{1}{2\pi i} \int_{\mu - \infty i}^{\infty i + \mu} \frac{\gamma e^{\tau p}}{(\gamma + p)p} F_1(p) dp, \tag{54}$$

for $S_r(\tau) \leq S \leq 1$ and,

$$U(S, \tau) = \frac{1}{2\pi i} \int_{\mu-\infty i}^{\infty i+\mu} \frac{\gamma e^{\tau p}}{(\gamma + p)p} F_2(p) dp, \tag{55}$$

for $S > 1$

In equation (54) and (55), $F_2(p)$ and $F_1(p)$ are determined by applying equations (51) to (53) on (41) to have

$$
\begin{aligned}
F_1(p) &= \frac{1}{2}\left(1 - \frac{b}{\sqrt{a^2+p}}\right) \cdot \left[1 - \frac{\gamma+p}{(b-\sqrt{a^2+p})\gamma}\right] \cdot S^{q_1} \\
&+ \frac{1}{2}\left(1 - \frac{b}{\sqrt{a^2+p}}\right) \cdot \left[1 - \frac{\gamma+p}{(\sqrt{p+a^2}+b)\gamma}\right]^{\frac{q_2}{q_1}} \cdot S^{q_2} - 1.
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
F_2(p) &= \left\{\frac{1}{2}\left(1 - \frac{m}{\sqrt{a^2+p}}\right) \cdot \left[1 - \frac{\gamma+p}{(m-\sqrt{a^2+p})\gamma}\right]\right. \\
&\left.+ \frac{1}{2}\left(1 - \frac{m}{\sqrt{a^2+p}}\right) \cdot \left[1 - \frac{\gamma+p}{(\sqrt{p+a^2}+m)\gamma}\right]^{\frac{q_2}{q_1}} - 1\right\} \cdot S^{q_2},
\end{aligned}
\tag{57}
$$

3.3 The American Options Partial Derivative Equation

For the calls American valuation, a transformation is introduced. The Black-Scholes price model of a deterministic asset paying dividend V yields D, interest rate r and volatility σ are illustrated as:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + SV_s(r - D) - Vr = 0 \tag{58}$$

Here, the call option underlying asset is denoted as S. The early American option exercise gives an optimal boundary exercise model, which is considered as a boundary free problem in the PDE setting. The domain $(0, \tilde{B}(t)) \times [0, T]$ for the equation (58) and the boundary constraints are taken as:

$$V(S, T) = S \in (0, \tilde{B}(T)), \max(S - K, 0), \tag{59}$$

$$\tilde{B}(T) = \max\left(K, \frac{Kr}{D}\right) \tag{60}$$

$$V(0, t) = 0, \tag{61}$$

$$V(\tilde{B}(t), t) = -K + \tilde{B}(t), \tag{62}$$

$$V_s(\tilde{B}(t), t) - 1 = 0, \tag{63}$$

3.4 American Call with Dividends

From the Black-Scholes equation, the American option value $C(S, t)$ satisfies

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + S(r - D_0)\frac{\partial C}{\partial S} - rC = 0, \tag{64}$$

This remains for non-optimal exercise. The payoff constraint is

$$\max(S - E, 0) = C(S, T) \tag{65}$$

Also, the option exercised can take place at any period; as such,

$$\max(S - E, 0) \leq C(S, t) \tag{66}$$

provided the boundary optimal exercise $S = S_r(t)$

$$C(S_r(t), t) = -E(S_r(t), t)\frac{\partial C}{\partial S} + S_r(t) = 1 \tag{67}$$

If the boundary optimal exercise occur therefore (64) is valid when $\max(S - E, 0) < C(S, t)$ due to the fact that $\max(S - E, 0)$ is not the Black-Scholes model solution. (64) can be replaced by an inequality

$$S(r - D_0)S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} - rC \leq 0 \tag{68}$$

The inequality is valid if $\max(S - E, 0) < C(S, t)$.

3.5 Call Dividends Analysis

Here, the simplification of dividend payments with Black-Scholes model is carried out by taking that the dividend payments and interest rate satisfy $0 < D_0 < r$. Therefore, equations (64) to (66) are transformed to dimensionless form by introducing the variables

$$S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad C(S, t) = Ec(x, \tau) - E + S \tag{69}$$

the result is

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (k' - 1)S\frac{\partial c}{\partial x} - kc + f(x) \tag{70}$$

where $k = \frac{r}{\frac{1}{2}\sigma^2}$, $k' = \frac{r - D_0}{\frac{1}{2}\sigma^2}$ for $-\infty < x, \infty$ and $\tau > 0$. The function $c(x, 0)$ gives the initial formation

$$\max(1 - e^x, 0) = c(x, 0) \tag{71}$$

The function is resulted from f

$$f(x) = k - (k - k')e^x \tag{72}$$

Taking that free boundary occur, $x = x_r(t)$, at the boundary to obtain

$$c(x_r(\tau), \tau) = (x_r(\tau), \tau)\frac{\partial c}{\partial \tau} \tag{73}$$

From equation (73), at expiration time gives

$$\frac{\partial c}{\partial \tau} = f(x) \tag{74}$$

For $f(x) > 0$, $x_0 > x > 0$, and c is positive. Give $x_0 < x$ when $f(x) < 0$, then c is negative; as such, when $x > 0$, $c > 0$ therefore constraint is not satisfied.

4 MONTE CARLO TECHNIQUE

The idea of Monte Carlo computation into finance was introduced by Boyle. The Monte Carlo numerical technique is used when closed-form solution is not available. The approach is a better pricing dependent path option. Monte Carlo computational scheme employes the valuation risk results; in risk-neutral world, expected payoff is computed by adopting a sampling technique. The major Monte Carlo simulation techniques are as follows:

- Compute an asset underlying path with the risk-neutralization constraint in between the time horizon.
- The payoff discounted is equivalent to the interest free-risk path rate.
- The approach is repeated for higher simulation numbers of sample path.
- The cash flow discounted average over the sample value option paths is determined.

A Monte Carlo computation uses approach for random sampling outcome processes according to the stock price [6]

$$dS = \mu S dt + \sigma S dW(t) \tag{75}$$

where stock price is S and Wiener process is dW_t. Given that δS is the stock price increment in the time interval of δt, then

$$\frac{\delta S}{S} = \mu \delta t + \sigma Z \sqrt{\delta t} \tag{76}$$

where $Z \sim N(0,1)$, expected risk-neutral return is μ and stock price volatility is σ, which can be expressed as

$$S(t + \delta t) - S(t) = \mu S(t)\delta t + \sigma S(t)Z\sqrt{\delta t} \tag{77}$$

At time $t + \delta t$ the value of S is computed taking from the S initial value, then at time $t + 2\delta t$ the value of S calculated and so on. A random N sample from a normal distribution for complete trial calculations is done in S. Its give better accurate simulation of $\ln S$ than S by using Ito's lemma to modify the asset price procedure

$$d(\ln S) = \sigma dW(t) - \left(\frac{\sigma^2}{2} - \mu \right) dt$$

so that

$$\ln(t + \delta t)S - \ln S(t) = \sigma Z\sqrt{\delta t} - \left(\frac{\sigma^2}{2} - \mu \right) \delta t$$

or

$$S(t + \delta t) = S(t) \exp\left[\sigma Z\sqrt{\delta t} - \left(\frac{\sigma^2}{2} - \mu \right) \delta t \right] \tag{78}$$

The Monte Carlo computation is specifically useful when there is payoff financial derivatives depending on the path and life asset underlying option. For instance, considering a maturity time T of Asian Stock price options defined as

$$S_T^j = S \exp\left[\sigma Z \sqrt{T} - \left(\frac{\sigma^2}{2} - \mu\right)T\right] (79) The evaluated Call Asian value option is C = \frac{1}{M}\sum_{j=1}^{M} e^{-Tr} \max[S_T^j - S_t, 0](80)$$

where trial numbers is M and $j = 1, 2, 3, ..., M$, this is an unbiased price derivative evaluation. When M is high, the limit central theorem gives an estimation realistic interval based on the discounted payoff of sample variance. If discounted payoffs mean is $\bar{\mu}$ and standard deviation is ω, then $\frac{\omega}{\sqrt{M}}$ is the estimated standard error. A price f confidence interval of 0.95% for the derivative is taken as:

$$\bar{\mu} - \frac{1.96\omega}{\sqrt{M}} < f < \bar{\mu} + \frac{1.96\omega}{\sqrt{M}} \tag{81}$$

with the normally distributed assumption for f.

5 COMPARISON OF SCIENTIFIC COMPUTATIONS

Scientific calculation is the science computing, and it is major aspect of this study. The interest on a relative balance between computer science and the modern and classical computational mathematics elements. To compare the results of the standard numerical approximations discussed above, we computed some examples of call options.

Example 1:

Depicts the two schemes performance against the put European Black-Scholes price with $K = 50, r = 0.05, \sigma = 0.25, T = 3$.

Table 1. A Verification of the put european black-scholes price

S	Black-Scholes	Monte-Carlo	Implicit Euler
10	33.0363	33.0345	33.0369
20	23.2276	23.2291	23.2300
30	14.7739	14.7748	14.7749
40	8.7338	8.7374	8.7348
50	4.9564	4.9559	4.9563
60	2.7621	2.7602	2.7612
70	1.5328	1.5324	1.5325
80	0.8538	0.8543	0.8537

Table 1 displays the option price variation with the asset price underlying, S. The outcomes illustrate that the two schemes are mutually consistent, effective, and agree with the values of Black-Scholes. Meanwhile, it may not be essential is applying such numerical schemes in practice where explicit formula exist.

6 RESULTS AND DISCUSSION

The study presents scientific computing for derivatives valuing where analytical solution do not exist. This adopted the uses of Monte Carlo numerical scheme for an American price option. The Monte Carlo

computation has to do with a simple random variable numbers with diverse paths that a derivative could take in a neutral-risk society. The free-risk-discounted interest rate and payoff are determined for each paths. The discounted payoffs average arithmetic is the derivative estimated value. The implicit finite implicit difference approach more complex but has good because the user do not need to take any particular caution for convergence.

In practice, the accuracy and derivative characteristics determined the applied evaluation technique to be adopted. The Monte Carlo computation works on the life derivative from the beginning to the end forwardly. It may be employed for the derivative of the European-style and can adapt to high complexity deals of the payoff. It is adaptively proficient as the underlying variable numbers is raised. Moreover, one major setbacks in utilizing Monte-Carlo computation is that it is complex to implement for early American-style, though it performs better for path-dependent European-style pricing.

7 CONCLUSION

This research provides a comprehensive analysis of the application of the Black-Scholes and Monte Carlo derivative model methods in the American options context. The analysis underscores the complexity inherent in the American option modeling, majorly as a result of their early exercise characteristics, which is not completely taking care by the Black-Scholes standard equation. Through an advanced Monte Carlo simulations, this research evaluates the performance and accuracy of this computational approaches in pricing American options. Key outcomes from the analysis can be summarized: The Black-Scholes model offers a foundational and relatively straightforward approach to option pricing, its assumptions and limitations become evident when applied to American options; Monte Carlo simulations, with their ability to handle a wide range of potential price paths and complex boundary conditions, demonstrate superior accuracy and flexibility; The comparative analysis highlights that although the Black-Scholes model remains a vital tool in financial engineering, Monte Carlo simulations provide a more robust and adaptable framework for American option pricing.

In summary, the research contributes valuable insights into the strengths and limitations of diverse scientific computing techniques in option pricing. It underscores the importance of selecting appropriate models and methods based on the specific characteristics of the financial instruments being analyzed. Future work could explore hybrid approaches that integrate the analytical strengths of the Black-Scholes equation with the computational precision of Monte Carlo simulations, potentially offering even more effective solutions for pricing American options.

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of manuscripts.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

[1] Breen R. The accelerated binomial option pricing model. J. of Fin. and Quan. Anal. 1991;26:153-164.

[2] Dar AA, Khan MS, Azad I, Jayaraman G, Farooqi AR. Exploring the impact of input variables on option value: A study using experimental design and analysis techniques. Discover Applied Sciences, 2024;6(1):28.

[3] McKean HP. Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics. Industrial Management Review. 1965;6:32-39.

[4] Dar AA, Anuradha N, Nihel Z. Comparison of European Option Pricing Models at Multiple Periods. In Handbook of Research on Engineering, Business, and Healthcare Applications of Data Science and Analytics. IGI Global, 2021;21:149-166.

[5] Geske R, Shastri K. Valuation by approximation: A comparison of alternative option valuation techniques. J. of Fin. Quan. Anal. 1985;20:45-71.

[6] MacMillan L. Analytic approximation for the American put option. Adv. in Futures and Options Res. 1986;1:119-139.

[7] Han H, Huang Z. A class of artificial boundary conditions for heat equation in unbounded

domains. Comput. Maths and Appl. 2002;43:889-900.

[8] Dar AA, Anuradha N. Use of Taguchi method for optimisation of process parameters of option pricing model. Int. J. of Ser. Eco. and Mgt., 2020;11(1):1-20.

[9] Brennan M, Schwartz E. The valuation of American put options. J. of Fin. 1977;32:449-462.

[10] Hull JC. Options, futures and other derivatives. Pearson Education Inc. Fifth Edition: Prentice Hall, New Jersey; 2003.

[11] Dar AA, Anuradha, N. Studies on European call option of binomial option pricing model using Taguchi's L27 orthogonal array. Int. J. of Intelligent Enter., 2020;7(1-3):234-249.

[12] Wu L, Kwok YK. A front-Fixing finite difference method for the valuation of American options. J. of Fin. Engin. 1997;6:83-97

[13] Michael CF, Scott B, Laprise B, Madan DB, Su Y, Wu R. Pricing American options: A comparison of Monte Carlo simulation approaches. J. of Comput. Fin. 2001;4(3):39-88.

[14] Brennan M, Schwartz E. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. J. Fin. Quan. Anal. 1978;13:461-474.

[15] Cox J, Ross S, Rubinstein M. Option pricing: A simplified approach. J. of Fin. Eco. 1979;7:229-263.

[16] Glasserman P. Monte Carlo Methods in Financial Engineering. Springer-Verlag: New York; 2004.

[17] Tilley AJ. Valuing American options in a path simulation model. Trans. of the Soc. of Actuaries. 1993;45:93-104.

[18] Brandimarte P. Numerical methods in finance and cconomics: A MATLAB-based introduction. Wiley-Interscience. 2006;6:346-356.

[19] Boyle P, Broadie M, Glasserman P. Monte Carlo methods for security pricing. J. of Eco. Dynamics and Control. 1997;21(8-9)1267-1321.

[20] Han H, Wu X. Approximation of infinite boundary conditions and its application to finite element method. J. of Comput. Maths, 1985;3:179-192.

[21] Wilmott P, Howison S, Dewynne J. Option pricing: Mathematical models and computation. Oxford Financial Press: UK; (1993).

[22] Geske R, Johnson H. The American put options valued analytically. J. of Fin. 1984;39:1511-1524.

[23] Han H, Wu X. A fast numerical method for the Black-Scholes equation of American options. J. on Numer. Anal. 2004;41:2081-2095.

[24] Karatzas I. On the pricing of the American option. Appl. of Mathematica and Optimization. 1988;17:37-60.

[25] Kemna AGZ, Vorst ACF. A pricing method for options based on average asset values. J. of Banking and Fin. 1990;14:113-129.

[26] Broadie M, Glasserman P. Pricing American-style securities using simulation. J. of Eco Dynamics and Control. 1997;21:1323-1352.