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Automorphism Groups of Regular Elements with Von-Neumann Inverses of Local Near-Rings Admitting Frobenius Derivations

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This paper presents the classification of the invariant subgroups of the automorphism groups of the regular elements obtained from finite local near-rings, the appropriate algebraic structure to study non-linear functions on finite groups. Just as rings of matrices operate on vector spaces, near-rings operate on groups. In this paper, we construct classes of zero symmetric local near-ring of characteristic p^k; $k = 1, 2$, $k \geq 3$ admitting frobenius derivations, characterize the structures of the cyclic groups generated by the regular elements $R(\mathcal{N})$ as well as the structures and the orders of the automorphism groups $Aut(R(\mathcal{N}))$ of the regular elements.

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1 Introduction

Near-rings with identity are the appropriate algebraic structures used to study non-linear functions on finite groups. Let $\mathcal{N}$ be a zero symmetric near-ring from the classes of near-rings considered in this paper. Suppose $R(\mathcal{N})$ is the set of all the regular elements in $\mathcal{N}$, then every element $x \in R(\mathcal{N})$ is such that $x = 0$ or $x \in \mathcal{N}^*$. The automorphism groups of $R(\mathcal{N})$ denoted $Aut(R(\mathcal{N}))$ is the set whose elements are the automorphisms $\sigma : R(\mathcal{N}) \to R(\mathcal{N})$ and where the group operation is composition of automorphisms. Thus the group structure of $Aut(R(\mathcal{N}))$ is obtained as a subgroup of the symmetric groups $Sym(R(\mathcal{N}))$.

Idealization modules, rings and near-rings is perhaps one of the new approaches in ring classification problems up to isomorphism. The works in[1] determined constructions of idealized completely primary finite rings of characteristic $p^n : n = 1, 2, 3$ and determined the structures of the unit groups R^*. This was however neither extended to the automorphisms of R^* or the Von-Neumann regular elements $VRN(R)$ nor to the dual results in near-rings. Osba, Henriksen and Osama [2] conducted a classification survey on combining local and Von-Neumann Regular Rings as a basis upon which the regularity properties of rings and their ideals could be explored. The rings studied in [2] were finite and their Von-Neumann inverses gave some asymptotic patterns. Their findings demonstrated how to combine the Von- Neumann inverses of classes of rings such as the power series rings and the ring of integers. The study was however not extended to the automorphisms of the said algebraic structures. In a closely related research, the study on regular elements of Galois rings can be attributed to Osama and Emad [3] where they characterized the regular elements in the ring of integers modulo n, $\mathbb{Z}_n$. Furthermore, they studied the arithmetic functions denoted as $V(n)$ and determined the relationship between $V(n)$ and the Euler's phi function, $\varphi(n)$. This gave an extension of the ring theoretic algebra employed in counting the regular elements of $\mathbb{Z}_n$ to the number theoretic methodologies. For instance, the research revealed that if a is a regular element in $\mathbb{Z}_n$, then $a^{(-1)} \equiv a^{\varphi(n)-1}$ $(mod\ n)$. They proposed a criterion for getting the possible Von Neumann inverses in the set of regular elements of $\mathbb{Z}_n$ and explored the asymptotic properties of $V(n)$. Their findings did not consider extensions and idealization using maximal submodules of $\mathbb{Z}_n \forall n \in \mathbb{Z}$. Closely related works can also be seen in Osba et al [4] and Oduor, Omamo and Musoga[5]. To obtain a classification of algebraic structures, it is imperative to obtain group structures that are isomorphic to those structures, the automorphism groups. Therefore, this paper gives a complete classification of the regular elements $R(\mathcal{N})$ of some classes of zero-symmetric local near-rings which admit two classes of derivations, that is a generalized frobenius derivation in Construction I and an identity frobenius automorphism in construction II.

2 Local Near-Ring of Characteristics p, p^2 and p^k : $k \geq 3$ Admitting Frobenius Derivations

2.1 Construction I

Let $R_0 = G\mathcal{N}(p^{kr}, p^k)$ be a Galois near-ring of order p^{kr} and characteristic p^k and let $\mathcal{M} = \langle u_i \rangle : i = 1, \cdots h$ be an $h-$dimensional near-module of R_0 so that the ordered pair $(\mathcal{N}, +) = (R_0 \oplus \mathcal{M}, +)$ is a group. On $\mathcal{N}$, let

$$p^k u_i \quad = \quad \prod_{i=1}^{k} u_i = 0$$

and $u_i r_0 = (r_0)^{d_i} u_i$ when $k = 1,\ 2$ where $r_0 \in R_0,\ k, r$ are invariants and d_i a Frobenius derivation associated with elements of $\mathcal{M}$ and given by; $d_i(u_i) = (u_i)^p$. Let $\mathcal{J}$ be a near-ideal of $\mathcal{M}$ satisfying the condition that whenever $u_i, u_j \in \mathcal{J}$, we have $u_i \circ u_j \in \mathcal{J}$ or $u_i \circ u_j = 0$. If λ_i are any units of R_0, then we can see that the elements of $\mathcal{N} = R_0 \oplus \mathcal{M}$ are of the form: $x = r_0 + \sum_{i=1}^{h} \lambda_i u_i$. In fact, if $x = r_0 + \sum_{i=1}^{h} \alpha_i u_i$ and $y = s_0 + \sum_{i=1}^{h} \beta_i u_i$ are any two elements of $\mathcal{N}$, then we have their product as:

$$\begin{aligned} x \cdot y &= \Big(r_0 + \sum_{i=1}^{h} \alpha_i u_i\Big) \cdot \Big(s_0 + \sum_{i=1}^{h} \beta_i u_i\Big) \\ &= r_0 s_0 + \sum_{i=1}^{h} \Big\{\beta_i (r_0 + p^k R_0)^{d_i} + \alpha_i (s_0 + p^k R_0)^{d_i}\Big\} u_i. \end{aligned}$$

It has been shown in[6] that $(\mathcal{N}, +, \cdot)$ with the product given in the construction is a left (respective right) local near-ring whose unique maximal ideal is the Jordan ideal, $J(\mathcal{N})$.

2.2 Construction II

Let $R_o = G\mathcal{N}(p^{kr}, p^k)$. Let $i = 1, ..., h$ and $u_i \in Z_L(\mathcal{N})$ and $\mathcal{M} =< u_i >$.

Then,

$$\mathcal{N} = R_o \oplus \mathcal{M} = R_o \oplus \sum_{i=1}^{h} (R_o/pR_o)^i$$

is a group with respect to addition.

On $\mathcal{N}$, let

$$(r_o, \overline{r}_1, ..., \overline{r}_h)(s_o, \overline{s}_1, ..., \overline{s}_h) = (r_o s_o, r_o \overline{s}_1 + \overline{r}_1 s_o, ..., r_o \overline{s}_h + \overline{r}_h s_o)^{\delta}$$

where δ is the identity Frobenius automorphism. The multiplication turns $\mathcal{N}$ into a local zero symmetric near-ring with identity $(1, \overline{0}, ..., \overline{0})$.

Indeed $\mathcal{N} = R_o \oplus \mathcal{M}$ is commutative since δ is the identity Frobenius automorphism.

Proposition 2.1. *Consider $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k)$ where $k \geq 3$. Then, char$\mathcal{N} = p^k$ and:*

(i). $Z_L(\mathcal{N}) = pR_o \oplus \sum_{i=1}^{h} (R_o/pR_o)^i$

(ii). $(Z_L(\mathcal{N}))^{k-1} = p^{k-1} R_o \neq (0)$

(iii). $(Z_L(\mathcal{N}))^{k} = (0)$.

Lemma 2.1. *Let $\mathcal{N} = G\mathcal{N}(p^{kr}, p^k) \oplus \mathcal{M}$ where p is prime k and r are positive integers and $\mathcal{M}$ is a h-dimensional module over $\mathcal{N}$. Then if $h = 0$,*

(i) $R(\mathcal{N}) \cong (1 + Z(\mathcal{N})) \cup \{0\}$ *and*

(ii) $\mid R(\mathcal{N}) \mid = (p^{(k-1)r})(p^r - 1) + 1$

Proof. Let $a \in R(\mathcal{N}) \cong (1 + Z(\mathcal{N}))$. Then a is invertible or 0. But $\mathcal{N}$ is local means that a is regular i.e. $a \in R(\mathcal{N})$.

Thus $R(\mathcal{N}) \subseteq [< a > \times 1 + Z(\mathcal{N}))] \cup \{0\}$...(i)

Conversely, let $a \in R(\mathcal{N})$. Then by definition $\exists$ an element $b \in R(\mathcal{N})$ such that $a = a^2 b \Rightarrow a(1 - ab) = 0$.

If $a \in (\mathcal{N}^*)$ then $1 - ab = 0 \Rightarrow ab = 1$.

Hence b is a Von Neumann inverse of a. If is not a member of $\mathcal{N}^*$ then ab is not a member of $\mathcal{N}^*$ but $ab = aabb = a^2b^2 = abab = (ab)^2$.

Since $\mathcal{N}$ commutes $\Rightarrow ab = (ab)^2 \Rightarrow ab(1 - ab) = 0$.

Now $\Rightarrow 1 - ab$ is a unit and $ab = 0$ so that $a = 0$ because b is its Von Neumann inverse.

$[\{< a > \times 1 + Z(\mathcal{N})\} \cup \{0\}] \subseteq R(\mathcal{N})$..(ii)

Combining (i) and (ii) gives

$$\begin{aligned} R(\mathcal{N}) &\cong [1 + Z(\mathcal{N})] \cup \{0\} \\ &= < a > \times [1 + Z(\mathcal{N})] \cup \{0\} \end{aligned}$$

Next,

$$\begin{aligned} \mathcal{N}^* &= (\mathcal{N}^*/1 + Z(\mathcal{N})) \times 1 + Z(\mathcal{N}) \\ &\cong < a > \times [1 + Z(\mathcal{N})] \\ &= \mathbb{Z}_{p^r-1} \times [1 + Z(\mathcal{N})] \end{aligned}$$

But

$$\begin{aligned} \mid [1 + Z(\mathcal{N})] \mid &= \mid Z(\mathcal{N}) \mid \\ &= p^{(k-1)r} \end{aligned}$$

Therefore $\mid \mathcal{N}^* \mid = (p^r - 1)(p^{(k-1)r})$

But $R(\mathcal{N}) = \mathcal{N}^* \cup \{0\}$ $\mid R(\mathcal{N}) \mid = (p^r - 1)(p^{(k-1)r}) + 1$ as required. □

Theorem 2.1. *Let $\mathcal{N}$ be the near-ring constructed in I and II and $R(\mathcal{N})$ be the set of all the regular elements. Then*

(i).

$$R(\mathcal{N}) = \begin{cases} \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h \cup \{0\}, & Char\mathcal{N} = p; \\ \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^{h+1} \cup \{0\}, & Char\mathcal{N} = p^2. \end{cases}$$

(ii).

$$R(\mathcal{N}) = \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}} \times \mathbb{Z}_{2^{k-1}}^{r-1} \times (\mathbb{Z}_2)^h \cup \{0\}, & p = 2; \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^k-1}^r \times (\mathbb{Z}_p^r)^h \cup \{0\}, & p \neq 2 : Char\mathcal{N} = p^k : k \geq 3. \end{cases}$$

Proof. Let $\tau_1, ..., \tau_r \in \mathbb{F}_q$ with $\tau_1 = 1$ such that $\overline{\tau}_1, ..., \overline{\tau}_r$ form a basis for $\mathbb{F}_q$ regarded as a vector space over its prime subnear-field $\mathbb{F}_q$ where $q = p^r$ for any prime p and a positive integer r.

(i) **[case 1.]** Let $char\mathcal{N} = p$

It can be observed that for every $l = 1, ..., r$ and $1 \leq i \leq h$, $1 + \tau_l u_i \in 1 + Z(\mathcal{N})$ and

$$\begin{aligned} (1 + \tau_l u_1)^p &= (1 + \tau_l u_1 + \tau_l u_2)^p \\ &= ... \\ &= (1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)^p \\ &= 1. \end{aligned}$$

That is, $y^p = 1 \ \forall \ y \in 1 + Z(\mathcal{N})$.

Now for the positive integers $a_{1l}, a_{2l}, ..., a_{hl}$ with $a_{1l} \leq p, a_{2l} \leq p, ..., a_{hl} \leq p$, we notice that

$$\prod_{l=1}^{r}\{(1+\tau_l u_1)^{a_{1l}}\} \quad . \quad \prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2)^{a_{1l}}\} \, . \\ \ldots \quad . \prod_{l=1}^{r}\{(1+\tau_l u_1+\tau_l u_2+\ldots+\tau_l u_h)^{a_{1l}}\} = 1$$

will imply that $a_{il} = p, \forall l = 1, ...r$ and $1 \leq i \leq h$.

Let

$$\begin{aligned} S_{1l} &= \{(1+\tau_l u_1)^{a_1} : a_1 = 1, ..., p\} \\ S_{2l} &= \{(1+\tau_l u_1+\tau_l u_2)^{a_2} : a_2 = 1, ..., p\} \\ &\vdots \\ S_{hl} &= \{(1+\tau_l u_1+\ldots+\tau_l u_h)^{a_h} : a_h = 1, ..., p\}. \end{aligned}$$

Then, $S_{1l}, S_{2l}, ..., S_{hl}$ are all cyclic subgroups of $1+Z(\mathcal{N})$ and they are each of order p hence $1+Z(\mathcal{N})$ is abelian and each element $a \in 1 + Z(R)$ is such that $a^p = 1$.

Now,

$$\prod_{l=1}^{r} |< 1+\tau_l u_1 >| \quad . \quad \prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2 >| \ldots$$

$$\prod_{l=1}^{r} |< 1+\tau_l u_1+\tau_l u_2+\ldots+\tau_l u_h >| \quad = \quad p^{hr}.$$

The intersection of any pair of the cyclic subgroups $S_{1l}, S_{2l}, ..., S_{hl}$ gives the identity group and the product of the hr subgroups $S_{1l}...S_{hl}$ exhausts $1 + Z(R)$.

But

$$R(\mathcal{N}) = < a > \rtimes (1+Z(\mathcal{N})) \cup \{0\} = \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h \cup \{0\}$$

such that $o(a) = p^r - 1$. This settles the **case 1.**

(ii.)[**case 2**] Let $p^2 = char\mathcal{N}$

For every $l = 1, ..., r, (1 + p\tau_l)^p = 1, (1 + \tau_l u_1)^{p^2} = 1, (1 + \tau_l u_1 + \tau_l u_2)^{p^2} = 1, ..., (1 + \tau_l u_1 + ... + \tau_l u_h)^{p^2} = 1)$. For positive integers $a_l, b_{1l}, ..., b_{hl}$ with $a_l \leq p, b_{1l} \leq p^2, ..., b_{hl} \leq p^2$. It is clear that

$$\prod_{l=1}^{r}(1+p\tau_l)^{a_l} \quad . \quad \prod_{l=1}^{r}(1+\tau_l u_1)^{b_{1l}} \ldots.. \\ \prod_{l=1}^{r}(1+\tau_l u_1+\tau_l u_2+\ldots+\tau_l u_h)^{b_{hl}} \quad = \quad 1$$

$\Rightarrow al = p, b_{1l} = p^2, ..., b_{hl} = p^2$ for every $L = 1, 2, ..., r$ and $1 \leq i \leq h$.

Set

$$\begin{aligned} T_l &= \{1 + p\tau_l)^a : a = 1, ..., p\} \\ S_{1l} &= \{1 + \tau_l u_1)^{b_1} : b_1 = 1, ..., p^2\} \\ &\vdots \\ S_{hl} &= \{(1 + \tau_l u_1 + \tau_l u_2 +, ..., +\tau_l u_h)^{b_h} : b_h = 1, ..., p^2\} \end{aligned}$$

$T_l, S_1, ..., S_h$ are all cyclic subgroups of the group $1 + Z(\mathcal{N})$ and they are of the orders indicated by their definitions.
Since

$$\prod_{l=1}^{r} |< 1 + p\tau_1 >| \ldots.. \prod_{l=1}^{r} |< 1 + \tau_l u_1 + ... + \tau_l u_h >|= p^{(2h+1)r},$$

and the intersection of any pair of the cyclic subgroups $T_l, ..., S_h$ gives an identity group, the product of the $(h + 1)r$ subgroups $T_l, S_1, ..., S_h$ is direct and exhausts $1 + Z(\mathcal{N})$. Thus according to case 1, we have

$$R(\mathcal{N}) =< \alpha > \times (1 + \mathbb{Z}) \cup \{0\}$$

$$R(\mathcal{N}) = \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^{h+1} \cup \{0\}$$

(iii) **[Case 3].** Let char $\mathcal{N} = p^k : k \geq 3$. We provide the general case using $p =$ odd.

Notice that every $l = 1, ..., r; (1 + p\tau_1)^{p^{k-1}} = 1$
$(1 + \tau_l u_1)^{p^k} = 1, ..., (1 + p\tau_L u_1 + \tau_l u_2 + ... + \tau_l u_n)^{p^k} = 1$.
Let $a_l, b_{1l}, ..., b_{hl} \in \mathbb{Z}^+$ with $a_l \leq p^{k-1}$, $b_{il} \leq p^k : 1 \leq i \leq h$. We notice that

$$\prod_{l=1}^{r} \{(1 + p\tau_L)^{a_L}\} \cdot \prod_{l=1}^{r} \{(1 + \tau_l u_1)^{b_{1l}}\} \cdot \prod_{l=1}^{r} \{(1 + \tau_l u_1 + \tau_l u_2 + ... + \tau_l u_h)\} = 1$$

which implies that $a_l = p^{k-1}$, $b_{1l} = p^k = \cdots = b_{hl} = p^k$. Set

$$\begin{aligned} T_l &= < \{(1 + p\tau_l)^a \mid a = 1, ..., p^{k-1}\} > \\ S_{1l} &= < \{(1 + \tau_l u_1)^{b_1} \mid b_1 = 1, \cdots, p^k\} > \\ &\vdots \\ S_{hl} &= < \{(1 + \tau_l u_1 + \cdots + \tau_l u_n)^{b_h} \mid b_h = 1, \cdots, p^k\} > \end{aligned}$$

The sets defined are all cyclic subgroups of the group $1+Z(\mathcal{N})$ and they are of the indicated orders. Furthermore, the intersection of any pair of the cyclic subgroups indicated gives an identity group and the product of the $(h + 1)r$ subgroups gives:

$| T_l \times S_{1L} \times S_{hl} |= p^{k((h+1)r)-1}$ exhausting $1 + Z(\mathcal{N})$.

Thus $1 + Z(\mathcal{N}) \cong \mathbb{Z}^r_{p^{k-1}} \times (\mathbb{Z}^r_p)^h$.

Therefore

$$\begin{aligned} R(\mathcal{N}) &=< \alpha > \times (1 + (Z(\mathcal{N}))) \cup \{0\} \\ &= \mathbb{Z}_{p^r-1} \times \mathbb{Z}^r_{p^k-1} \times (\mathbb{Z}^r_p)^h \cup \{0\}. \end{aligned}$$

□

Theorem 2.2. *Let $\mathcal{N} = R_o \oplus \mathcal{M}$ where $r = 1$ and p-prime, $k \in \mathbb{Z}^+$. If $\mathcal{M} = R_0/pR_0 \oplus ... \oplus R_0/pR_0$. Let $r_0 \in R(R_0)$ then, its Von-Neumann inverse is*
$r_0^{-1} = r_0^{p^k-p^{k-1}-1}$ and $(r_0, ..., r_h)^{-1} = (r^{p^k-p^{k-1}-1}, -r_1 t_0 r_0^{-1}, ..., -r_h t_0 r_0^{-1})$

Theorem 2.3. *Let $\mathcal{N}$ be a near-ring of construction I and II, $R(\mathcal{N})$ be the set of all the regular elements including 0. Then if*
$Aut : R(\mathcal{N}) \rightarrow R(\mathcal{N})$ is an automorphism:

(i) when $char\mathcal{N} = p$, then $Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{hr}(G\mathcal{N}(p^r, p))] \cup \triangle$ where $\triangle = x \in R(\mathcal{N}) : Aut(x) = 0$

(ii) when $char\mathcal{N} = p^2$, then, $Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{(h+1)r}(G\mathcal{N}(p^{2r}, p^2))] \cup \triangle$

(iii) when
$char\mathcal{N} = p^k : k \geq 3$, then,
$Aut(R(\mathcal{N})) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{(k-1)r}(GN(p^{kr}, p^k))] \times GL_{hr}(GN(p^{kr}, p^k))] \cup \triangle$.

Proof. Proof of (i.) By enumeration, $R(\mathcal{N}) =< a > \times (1 + Z_L(\mathcal{N})) \cup \{0\}$ where
$< a >= \mathbb{Z}_{p^r-1}$. Since $gcd(p^r - 1, \mid 1 + Z_L(\mathcal{N}) \mid) = 1$, we have that

$$Aut(\mathbb{Z}_{p^r-1} \times 1 + Z_L(\mathcal{N})) \cong Aut(\mathbb{Z}_{p^r-1} \times Aut(1 + Z(\mathcal{N}))$$

But,

$$Aut(\mathbb{Z}_{p^r-1}) = (\mathbb{Z}_{p^r-1})^*$$

which is a permutation group whose order coincides with the order of $1 + Z_L(\mathcal{N})$.

Next, define a zero automorphism to be the set

$$\triangle = \{x \in R(\mathcal{N})\} : Aut(x) = 0.$$

Then clearly $\triangle = \{Aut(0) = 0_n\}$.

When $char\mathcal{N} = p$,

$$R(\mathcal{N}) = [\mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^h] \cup \{0\}$$

which implies that

$$AutR(\mathcal{N}) \cong [(\mathbb{Z}_{p^r-1})^* \times GL_{hr}(\mathbb{F}_p)] \cup \triangle.$$

This proves (i). The conditions (ii) and (iii) follow from the proof of (i) with modifications. ☐

Theorem 2.4. *Let $\mathcal{N}$ be a zero symmetric local near-rings from the class of near-rings of constructions I and II. Then:*

(i)

$$\mid Aut(R(\mathcal{N})) \mid \quad = \quad [\varphi(p^r - 1) \cdot \prod_{k=1}^{hr} (p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p$

(ii)

$$\mid Aut(R(\mathcal{N})) \mid \quad = \quad [\varphi(p^r - 1) \cdot \prod_{k=1}^{(h+1)r} (p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p^2$

(iii)

$$\mid Aut(R(\mathcal{N})) \mid \quad = \quad [\varphi(p^r - 1) \cdot \prod_{k=1}^{(k-1)r} (p^k - p^{k-1}) \cdot \prod_{k=1}^{hr} (p^k - p^{k-1})] + 1$$

when $char\mathcal{N} = p^k : k \geq 3$

Proof. (i) Let $char\mathcal{N} = p$

By definition of $\varphi(n)$ attributed to Osama and Emad [[3]],

$$|\ (\mathbb{Z}_{p^r-1})^* \ |= \varphi(p^r - 1)$$

and since $Aut(\mathbb{Z}_{p^r-1}) =|\ (\mathbb{Z}_{p^r-1})^* \ |= \varphi(p^r - 1)$ the prefix of right hand side to (i) is clear.

From the previous theorem, $Aut(1 + Z_L(\mathcal{N}) = GL_{hr}(\mathbb{Z}_p)$. Thus, we need to find all the elements of

$$GL_{hr}(\mathbb{Z}_p)$$

in the endomorphism, $End(1 + Z_L(\mathcal{N}))$ and calculate the distinct ways of extending such an element to an endomorphism. So we need all such matrices that are invertible modulo p.

Let $R_p \in End(1 + Z_L(\mathcal{N}))$ then the number of matrices $A \in R_p$ that are invertible modulo p are upper block triangular matrices whose number can be given as:

$$\sharp A \quad = \quad \prod_{k=1}^{n}(p^k - p^{k-1})$$

Now when $charR = p$, $n = hr$ therefore

$$\sharp A \quad = \quad \prod_{k=1}^{hr}[p^k - p^{k-1}]$$

this means that

$$|\ \mathbb{Z}_{p^r-1} \times GL_{hr}(\mathbb{Z}_p))\ | \quad = \quad \varphi(p^r - 1) \centerdot \prod_{k=1}^{hr}[p^k - p^{k-1}]$$

Finally $0 \in R(\mathcal{N})$ and $Aut(0) = 0$. Now $|\ Aut(0)\ |=|\ \{0\}\ |= 1$ thus

$$\begin{aligned} |\ AutR(\mathcal{N})\ | \quad &= \quad |\ [Aut(Z_{p^r-1}) \centerdot Aut(GL_{hr}(\mathbb{Z}_p))]\ | + |\ Aut\{0\}\ | \\ &= \quad [\varphi(p^r - 1) \centerdot \prod_{k=1}^{hr}(p^k - p^{k-1})] + 1 \end{aligned}$$

as required. The proofs to (ii) and (iii) are similar to proofs of (i) with modifications on the orders of $GL_n(\mathbb{Z}_p)$ □

3 Conclusion

This study has determined the structures of the regular elements $R(\mathcal{N})$ of idealized local near-rings with derivations as cyclic groups using the Fundamental Theorem of Finitely generated Abelian Groups. The algebraic structures of $R(\mathcal{N})$ have been completely classified as subgroups of $Sym(\mathcal{N}^*)$ using the automorphisms groups $Aut(R(\mathcal{N}))$. The research reveals unique algebraic structures.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Oduor MO, Ojiema MO, Eliud M. Units of commutative completely primary finite rings of characteristic pn. International Journ. of Algebra. 2013;7(6):259-266.

[2] Osba A, Henriksen M, Osama A. Combining Local and Von-Neumann Regular Rings. Comm. Algebra. 2004;32:2639 - 2653.

[3] Osama A, Emad AO. On the regular elements in $\mathbb{Z}_n$, Turk J. Math. 2008;32:31-39.

[4] Osba A, Henriksen M, Osama A, Smith F. The Maximal Regular Ideals of some commutative Rings, Comment. Math. Univ. Carolinea. 2006;47(1):1-10.

[5] Oduor MO, Omamo AL, Musoga C. On the Regular Elements of Rings in which the product of any two zero divisors lies in the Galois subring. IJPAM. 2013;86:7-18.

[6] Ojiema M. Onyango, Onyango B. Achieng, Abuga J. Motanya. On a class of idealized near-rings admitting frobenius derivations. Journal of Advances in Mathematics and Computer Science. 2013;35(5):1-9.

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