Asian Research Journal of Mathematics

Volume 19, Issue 5, Page 41-50, 2023; Article no.ARJOM.97799

ISSN: 2456-477X

Bipartite Domination Number of Mycielski Graph of Some Graph Families

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2023/v19i5658

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/97799

Received: 15/01/2023

Accepted: 20/03/2023

Published: 24/03/2023

Original Research Article

Abstract

For a nontrivial connected graph G, a non-empty set $S \subseteq V(G)$ is a bipartite dominating set of graph G, if the subgraph $G[S]$ induced by S is bipartite and for every vertex not in S is dominated by any vertex in S. The bipartite domination number denoted by $\gamma_{bip}(G)$ of graph G is the minimum cardinality of a bipartite dominating set G. In this paper, we determine the exact bipartite domination number of a crown graph and its mycielski graph as well as the bipartite domination number of the mycielski graph of path and cycle graphs.

Keywords: Bipartite dominating set; bipartite domination number.

2020 Mathematics Subject Classification: 05C35.

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1 Introduction

One of the fast-developing fields in graph theory is the study of domination and its related topics. For instance, Bachstein et al.,[1] published a study about the bipartite domination in graphs, and they introduced and defined the concept of bipartite dominating set and bipartite domination number. They investigated the case that S must induce a bipartite subgraph. Their idea about the bipartite domination came from the published study entitled "Bipartite domination and simultaneous matroid covers" [2].

Recently, we published a new parameter in bipartite domination in graphs entitled Bipartite Domination in Some Classes of Graphs [3], which was inspired by the work of Bachstein et al.,[1]. It presented new approach in finding the minimum cardinality of a bipartite dominating sets of path and cycle graphs via congruence modulo. Moreover, possible exact values of the bipartite domination number of some classes of graph are generated.

In this paper, we extended the study of the bipartite domination number in crown graph up to its mycielski graph, including the mycielski graph of path and cycle.

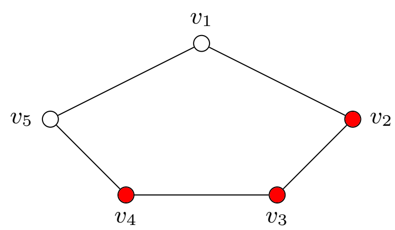
2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [1], [4], [5], [6], [3], [7].

Definition 2.1. [8] **(Total Dominating Set, Total Domination number)** A set $D \subseteq V$ is a total dominating set if every vertex $v \in V$ has, at least, a neighbor in D. The total domination number, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets.

Example 2.1. Consider the cycle graph C_5 below.

C_5 :



In the cycle graph illustrated above, the set $T = \{v_2, v_3, v_4\}$ is a total dominating set and that T is the minimum total domination number of C_5. Thus, $\gamma_t(C_5) = 3$.

Definition 2.2. [4] A graph $G = (V(G), E(G))$ is ***bipartite*** if $V(G)$ can be partitioned into two sets U and W (called ***partite sets***) so that every edge of G joins a vertex of U and a vertex of W.

In this study, we preserved the uniqueness of the partitioning of the bipartite graph, which means that we only consider the bipartite graph with no isolated vertices.

Definition 2.3. [5] **((Mycielski Graph)** Consider a graph G with $V(G) = \{v_1, v_2, v_3, ..., v_n\}$. Apply the following steps to the graph G:

i. Take the set of new vertices $U = \{u_1, u_2, u_3, ..., u_n\}$ and add edges from each vertex u_i of U to the vertices v_j if the corresponding vertex v_i is adjacent to v_j in G.

ii. Take another new vertex w_0 and add edges joining each element in U.

Here, the new graph obtained is the Mycielski graph, denoted by $\mu(G)$ of graph G.

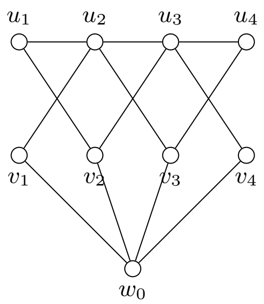
By the definition of Mycielski Graph, its vertex set can be express as follows:

$$V(\mu(G)) = V(G) \cup G' \cup \{w_0\}$$

where G' is the set of vertices that are vertex copy of each vertices in G and the set $\{w_0\}$ for which $deg_{\mu(G)}(w_0) = n-1$ for $|G| = n$. Also, if $\emptyset \neq S \subseteq V(G)$ then $S' \subseteq G'$ where S' is the set of vertex that are vertex copy of each vertex in S.

Example 2.2. Consider the mycielski graph of a path graph P_4 below:

$\mu(P_4)$:



Observe that $V(\mu(P_4)) = V(P_4) \cup P' \cup \{w_0\}$ where $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $P' = \{v_1, v_2, v_3, v_4\}$.

Definition 2.4. [5] **(Crown Graph)** A crown graph $G(n,n)$ is a graph on $2n$ vertices with two sets of vertices u_i and v_j and with an edge from u_i to v_j whenever $i \neq j$.

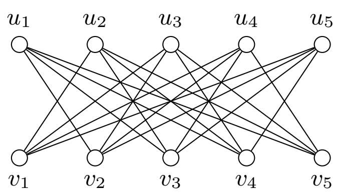
By the definition of Crown Graph, we can observe the following:

$$V(G(n,n)) = V(X) \cup V(Y)$$

such that X and Y are a partite set with $|X| = n = |Y|$.

Example 2.3. Consider the crown graph $G(5,5)$ below:

$G(5,5)$:

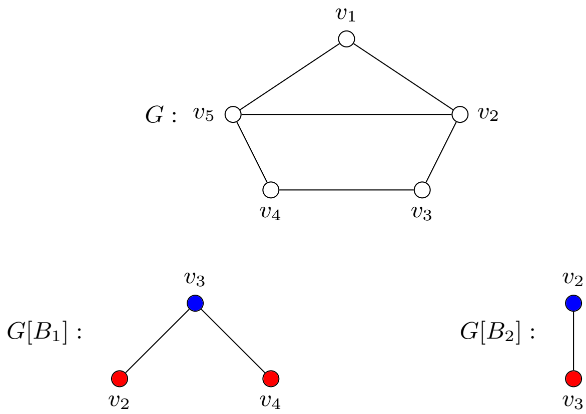


Notice that $V(G(5,5)) = X \cup Y$ where $X = \{u_1, u_2, u_3, u_4, u_5\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5\}$.

Definition 2.5. [3] **(Bipartite Dominating Set, Bipartite Domination number)**

A dominating set S of a graph G is a bipartite dominating set if the induced subgraph S, $G[S]$, is bipartite graph. The minimum cardinality of a bipartite dominating set is called bipartite domination number of G, denoted by $\gamma_{bip}(G)$. A nonempty set $S \subseteq V(G)$ whose cardinality is the $\gamma_{bip}(G)$ is called γ_{bip}-set of G.

Example 2.4. Consider the graph G below.



The possible bipartite dominating sets for the graph G are $B_1 = \{v_2, v_3, v_4\}$ and $B_2 = \{v_2, v_3\}$, where B_1 can be partitioned into partite sets $U = \{v_3\}$ and $W = \{v_2, v_4\}$, and B_2 can be partitioned into partite sets $U = \{v_2\}$ and $W = \{v_3\}$. Notice that B_2 consist the minimum cardinality of a bipartite dominating set. Thus, $\gamma_{bip} = 2$.

The following known result characterized the bipartite graph and will be used in the succeeding results.

Theorem. [9] A graph G is a bipartite graph if and only if it is 2-colorable.

3 Main Results

In this section, the bipartite domination number of crown graph and Mycielski graph of path graph, cycle graph, and crown graph are determined[10-17].

3.1 Bipartite Domination Number of a Crown Graph, $G(n, n)$ and its Mycielski Graph, $\mu(G(n, n))$

Theorem 3.1. *Let $G(n, n)$ be a crown graph with $V(G(n, n)) = V(X) \cup V(Y)$ as define. Then $B \subseteq V(G(n, n))$ is a bipartite dominating set if and only if $B = B_1 \cup B_2$ where $B_1 \subseteq V(X)$ and $B_2 \subseteq V(Y)$ such that $|B_1| \geq 2$ and $|B_1| \geq 2$.*

Proof. Suppose B is a bipartite dominating set of $V(G(n, n))$. Let $B = B_1 \cup B_2 \subseteq V(G(n, n))$ where $B_1 \subseteq V(X)$ and $B_2 \subseteq V(Y)$. If $|B_1| < 2$ and $|B_2| < 2$, then B is either a dominating set and $G(n, n)[B]$ is not a bipartite graph or $G(n, n)[B]$ is a bipartite graph and B is not a dominating set. Hence, in both cases, B is not a bipartite dominating set. A contradiction to the assumption that B is a bipartite dominating set of $G(n, n)$. Thus, $|B_1| \geq 2$ and $|B_2| \geq 2$.

Conversely, suppose $B = B_1 \cup B_2 \subseteq V(G(n, n))$ where $B_1 \subseteq V(X)$ and $B_2 \subseteq V(Y)$ such that $|B_1| \geq 2$ and $|B_1| \geq 2$. Then, $N[B] = N[B_1] \cup N[B_2] = X \cup Y = V(G(n, n))$. Thus, B is a dominating set in $G(n, n)$. Figuratively, one can easily see that $G(n, n)[B]$ is a bipartite graph. Thus, B is a bipartite dominating set in $G(n, n)$.

□

Corollary 3.2. *For the crown graph $G(n, n)$ the $\gamma_{bip}(G(n, n)) = 4$.*

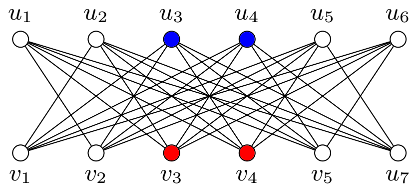
Proof. This is immediate from Theorem 3.1, by setting $|B_1| = 2$ and $|B_2| = 2$.

□

To illustrate the above results, we have the following example.

Example 3.5: Consider the crown graph $G(6,6)$ below:

$G(6,6)$:

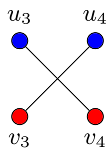


Observe that $V(G(6,6)) = X \cup Y$ where $X = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. By setting $|B_1| = 2$ and $|B_2| = 2$ where $B_1 \subseteq V(X)$ and $B_2 \subseteq V(Y)$. In this case, $B = B_1 \cup B_2$ such that $B_1 = \{u_3, u_4\}$ and $B_2 = \{v_3, v_4\}$. It can be easily seen that

$$\begin{aligned} N[B] &= N[B_1] \cup N[B_2] \\ &= \{v_1, v_2, v_3, v_4, v_5, v_6\} \cup \{u_3, u_4\} \cup \{u_1, u_2, u_3, u_4, u_5, u_6\} \cup \{v_3, v_4\} \\ &= V(G(6,6)). \end{aligned}$$

Thus, B is dominating set. Now, it can be seen that $G(6,6)[B]$ is a bipartite graph, shown below:

$G[B]$:



Hence, B is a bipartite dominating set of $G(6,6)$ and that B is the minimum bipartite domination number of $G(6,6)$. Thus, $\gamma_{bip}(G(6,6)) = 4$.

Theorem 3.3. *Let $G(n,n)$ be a crown graph and $B \subseteq V(G(n,n))$ be a bipartite dominating set of $G(n,n)$. Then, the set $B' \cup \{w_0\}$ is a bipartite dominating set of $\mu(G(n,n))$.*

Proof. Let $B \subseteq V(G(n,n))$ be a bipartite dominating set of $G(n,n)$. Set $S \subseteq V(\mu(G(n,n)))$ to be $S = B' \cup \{w_0\}$. Then,

$$\begin{aligned} N[S] &= N[B'] \cup N[\{w_0\}] \\ &= [B' \cup V(G(n,n)) \cup \{x\}] \cup [G'(n,n) \cup \{w_0\}] \\ &= V(G(n,n)) \cup G'(n,n) \cup \{w_0\} \\ &= V(\mu(G(n,n))). \end{aligned}$$

Thus, $S = B' \cup \{w_0\}$ is a dominating set of $\mu(G(n,n))$. Clearly, $\mu(G(n,n))[S] = \mu(G(n,n))[B' \cup \{w_0\}]$ is a bipartite graph. Hence, S is bipartite dominating set of $\mu(G(n,n))$.

□

The next results show the bipartite dominating set in the Mycielski graph of the crown graph and its bipartite domination number.

Theorem 3.4. *Let $G(n,n)$ be a crown graph. Then, $\gamma_{bip}(\mu(G(n,n))) = 5$.*

Proof. Let $B \subseteq V(G(n,n))$ be the minimum bipartite dominating set of $G(n,n)$ and let S_0 be the $\gamma_{bip} - set$ of $\mu(G(n,n))$. Choose $S = B' \cup \{w_0\}$. Then, by Theorem 3.4, S is a bipartite dominating set. Thus, $|S_0| \leq |S| = |B' \cup \{w_0\}| = |B'| + |\{w_0\}| = 4 + 1 = 5$. It remains to show that $|S_0| \geq |S|$.

On the contrary, suppose $|S_0| < 5$. Then, either S_0 is a dominating set of $\mu(G(n,n))$ or not a dominating set of $\mu(G(n,n))$. If S_0 is not a dominating set of $\mu(G(n,n))$, then we are done. Now, suppose S_0 is a dominating set. Then, $\mu(G(n,n))[S_0]$ is either 1-colorable or 2-colorable. If $\mu(G(n,n))[S_0]$ is 1-colorable then $\mu(G(n,n))[S_0]$ is not a bipartite graph. Now, if $\mu(G(n,n))[S_0]$ is 2-colorable, then $\mu(G(n,n))[S_0]$ contains isolated vertices. Hence, the uniqueness of the partitioning is not satisfied. Thus, $\mu(G(n,n))[S_0]$ is not a bipartite graph and that S_0 is not a bipartite dominating set of $\mu(G(n,n))$. Hence, $|S_0| \geq 5 = |S|$. Therefore, $\gamma_{bip}(\mu(G(n,n))) = |S_0| = 5$. □

3.2 Bipartite Domination Number of the Mycielski Graph of Path and Cycle Graph

Theorem 3.5. *Let G be a graph with $\Delta(G) = 2$. If $\emptyset \neq S \subseteq V(G)$ is a total dominating set of $V(G)$. Then the set $S' \cup \{w_0\} = B \subseteq V(\mu(G))$ is a bipartite dominating set of $\mu(G)$.*

Proof. Let $S \subseteq V(G)$ be a total dominating set in G and let S' be the copy of S in G'. Suppose $B = S' \cup \{w_0\} \subseteq V(G)$. Then,

$$\begin{aligned} N[B] &= N[S' \cup \{w_0\}] \\ &= N_{\mu(G)}[S'] \cup N_{\mu(G)}[\{w_0\}] \\ &= (S' \cup \{w_0\} \cup V(G)) \cup (\{w_0\} \cup G') \\ &= V(G) \cup G' \cup \{w_0\} \\ &= V(\mu(G)). \end{aligned}$$

Thus, B is a dominating set. Clearly, $\mu(G)[B]$ is a bipartite graph. Therefore, B is a bipartite dominating set of $\mu(G)$.

□

In [8], we have the following results for the total domination number which will be used in the next result.

Theorem 2.9.[8] For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$. In other words,

$$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \textit{if } n \equiv 0 \pmod 4 \\ \frac{n+1}{2} & \textit{if } n \equiv 1, 3 \pmod 4 \\ \frac{n+2}{2} & \textit{if } n \equiv 2 \pmod 4 \end{cases}$$

Theorem 3.6. *Let P_n and C_m be the path graph and cycle graph, respectively, with $n \geq 2$ and $m \geq 3$. Then,*

$$\gamma_{bip}(\mu(P_n)) = \gamma_{bip}(\mu(C_m)) = \begin{cases} \frac{n+2}{2} & \textit{if } n \equiv 0 \pmod 4 \\ \frac{n+3}{2} & \textit{if } n \equiv 1, 3 \pmod 4 \\ \frac{n+4}{2} & \textit{if } n \equiv 2 \pmod 4 \end{cases}$$

Proof. For the Path graph P_n. Let $V(P_n) = \{v_1, ..., v_n\}$. Then, $V(\mu(P_n)) = V(P_n) \cup P' \cup \{w_0\}$ where P' is the set of vertices that are vertex copy of each vertex in P_n. Let $B \subseteq V(\mu(P_n))$ be γ_{bip}-set of $\mu(P_n)$. Suppose S is a γ_t-set of P_n. Then, by Theorem 3.5, $B_0 = S' \cup \{w_0\}$ is a bipartite dominating set of $\mu(P_n)$. Thus, $|B| \leq |B_0|$. By Theorem 2.9 [8],

$$|B| \leq |B_0| = |S'| + |\{w_0\}| = \gamma_t(P_n) + 1 = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+3}{2} & \text{if } n \equiv 1, 3 \pmod 4 \\ \frac{n+4}{2} & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

To show that $|B| \geq |B_0|$. Suppose $|B| < |B_0|$. Then, either B is a dominating set of $\mu(P_n)$ or not a dominating set of $\mu(P_n)$. If B is not a dominating set of $\mu(P_n)$ we are done. Now, suppose B is a dominating set, then $\mu(P_n)[B]$ is a 1-colorable graph. Thus, $\mu(P_n)[B]$ is not a bipartite graph. Hence, B is not a bipartite dominating set. A contradiction to the assumption at B. Hence, $|B| \geq |B_0|$. Therefore,

$$\gamma_{bip}(P_n) = |B| = |B_0| = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n+3}{2} & \text{if } n \equiv 1, 3 \pmod 4 \\ \frac{n+4}{2} & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

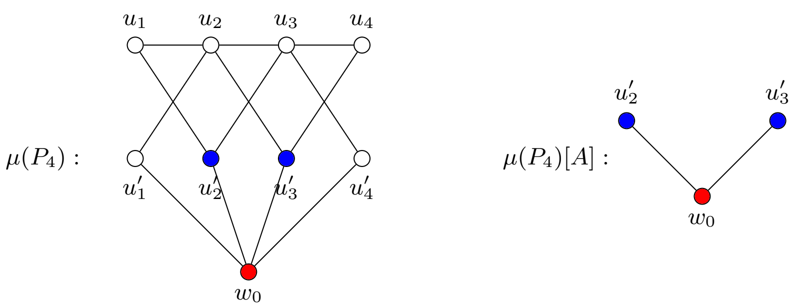
Similarly, for the Cycle graph C_m, $m \geq 3$.

□

To illustrate Theorem 3.6, we have the following examples.

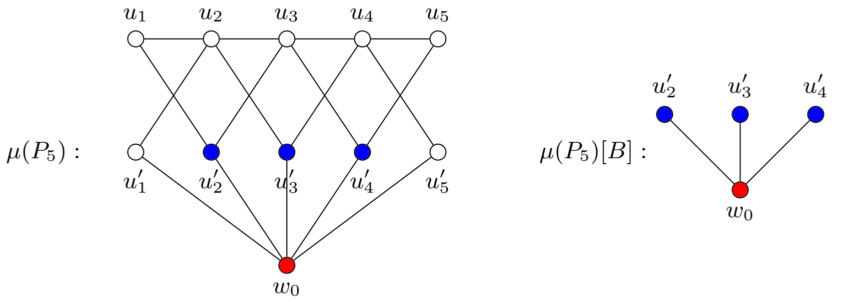
Example 3.6. Consider the following Mycielski graph of path graph $\mu(P_n)$.

For $n \equiv 0 \pmod 4$, choose $\mu(P_4)$, illustrated below.



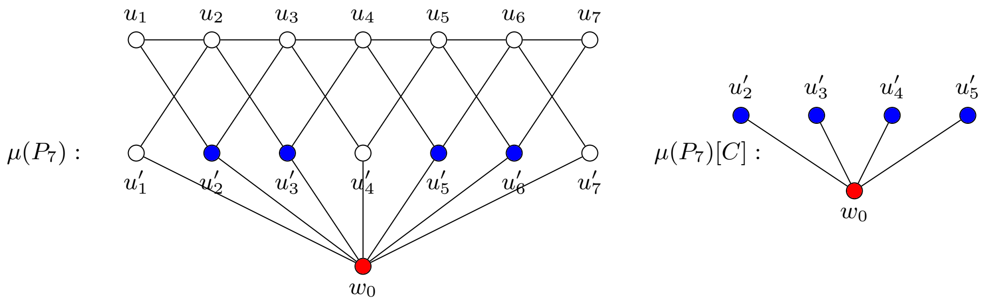
The set $A = \{u'_2, u'_3, w_0\} \subseteq \mu(P_4)$. Clearly, $\mu(P_4)[A]$ is a bipartite graph of $\mu(P_4)$. Thus, A is a bipartite dominating set and also the minimum bipartite dominating set in $\mu(P_4)$. Hence, $\gamma_{bip}\mu(P_4) = |A| = \frac{4}{2} + 1 = 3$.

For $n \equiv 1 \pmod 4$, choose $\mu(P_5)$, illustrated below.



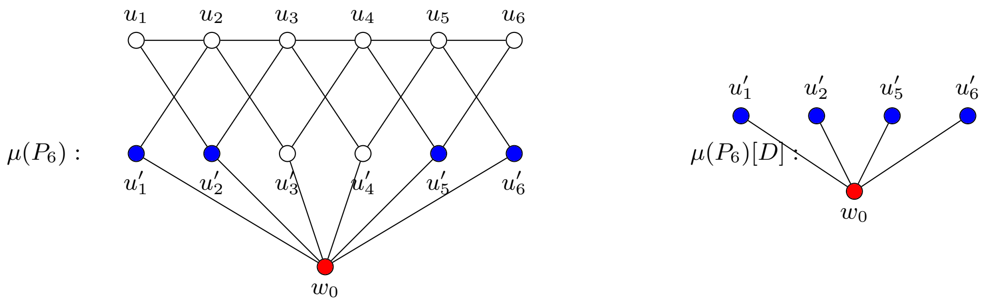
The set $B = \{u'_2, u'_3, u'_4 w_0\} \subseteq \mu(P_5)$. Clearly, $\mu(P_5)[B]$ is a bipartite graph of $\mu(P_5)$. Thus, B is a bipartite dominating set and also the minimum bipartite dominating set $\mu(P_5)$. Hence, $\gamma_{bip}\mu(P_5) = |B| = \frac{5+1}{2} + 1 = 4$.

For $n \equiv 3(\text{mod } 4)$, choose $\mu(P_7)$, illustrated below.



The set $C = \{u'_2, u'_3, u'_4, u'_5, w_0\} \subseteq \mu(P_7)$. Clearly, $\mu(P_7)[C]$ is a bipartite graph of $\mu(P_7)$. Thus, C is a bipartite dominating set and also the minimum bipartite dominating set $\mu(P_7)$. Hence, $\gamma_{bip}\mu(P_7) = |C| = \frac{7+1}{2} + 1 = 5$.

For $n \equiv 2(\text{mod } 4)$, choose $\mu(P_6)$, illustrated below.



The set $D = \{u'_1, u'_2, u'_5, u'_6, w_0\} \subseteq \mu(P_6)$. Clearly, $\mu(P_6)[D]$ is a bipartite graph of $\mu(P_6)$. Thus, D is a bipartite dominating set and also the minimum bipartite dominating set $\mu(P_6)$. Hence, $\gamma_{bip}\mu(P_6) = |D| = \frac{6+2}{2} + 1 = 5$.

4 Conclusion

This paper presents the findings of the bipartite domination number for both crown graph and its Mycielski graph, as well as the Mycielski graph of path and cycle graphs. Additionally, the necessary conditions for a bipartite dominating set on these graphs have been obtained. Moving forward, there is ample opportunity for further investigation into various graph families in conjunction with their corresponding Myscielski graph. It would be intriguing to explore additional results in this area.

Acknowledgment

The authors would like to thank the anonymous referees for helpful and valuable comments.

Competing Interests

The authors declare that they have no competing interests.

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