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Existence of Fixed Points for a Class of Contractive Maps in 2-metric Spaces

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Authors' contributions

This work was carried out in collaboration between the authors. Author PD surveyed the literatures and posed problems with an analysis of findings in writing a draft form of the manuscript. Author MS managed the literatures and analyzed the statements of the theorems and their proofs in detail by citing suitable examples in support of the theorems or to examine the validity of the conditions assumed in theorems. Finally the authors read and approved the final manuscript.

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ABSTRACT

In a setting of 2-metric space we have proved some existence theorems for fixed points of (ϕ,ψ) contractive maps and (ϕ,k) contractive maps which are certain generalization of contractive maps existing in the literatures.

Keywords: 2-metric space; (ϕ,ψ) contractive maps; (ϕ,k) contractive maps; fixed point.

1. INTRODUCTION

The concept of 2-metric space and its subsequent study could be found in a series of papers of S. Gähler [1- 3]. The analogue study of Banach contraction Theorem on a 2-metric space was initiated by Iseki et al. [4] and Iseki [5]. They proved that a Banach type contraction maps on a bounded complete 2-metric space possesses a unique fixed point. Existence of

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fixed points and common fixed points of different kinds of contractive mappings over a 2-metric space has been examined by Iseki [5], Rhoades [6], Saha and Mukherjee [7], Saha and Dey [8,9], Saha et al. [10,11], Naidu and Prasad [12] and by many others. Recently many generalizations of the contractive maps dealt with fixed points in a setting of metric spaces as well as 2-metric space have been proved by weakening the hypothesis retaining the convergence property of the successive iterates of maps. One such theorem has been proved by Boyd and Wong [13] in metric spaces. In this paper we have introduced a certain class of contractive operators namely (ϕ,φ) contractive operators and (ϕ,k) contractive operators that ensure the existence of fixed points in a 2-metric space. Here in light of Banach [14] and Kannan [15] we have been able to prove some fixed point theorems in two sections for a class of (ϕ,φ) contractive maps and (ϕ,k) contractive maps respectively in a setting of 2-metric space. Examples have been cited either to support our theorems or to examine the validity of the conditions assumed in theorems.

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Before going to our main results we recall the following basic definitions and supporting lemma.

Section-1

PRELIMINARIES

Definition 1.1

Let *X* be a non empty set. A non negative real valued function $d: X\times X\times X\to\mathbb{R}$ is said to be a 2-metric if the following conditions hold:

(i) For two distinct elements x, y of X, there exist an element z of X such that $d(x,y,z)\neq 0$

(ii) $d(x,y,z)=0$, when at least two of $x,y,z\in X$ are equal

(iii) $d(x,y,z)=d(x,z,y)=d(z,y,x)$ for all *x*, *y*, *z* in X and

(iv) $d(x,y,z)\le d(x,y,w)+d(x,w,z)+d(w,y,z)$ for all $x,y,z,w\in X$.

If *d* is a 2-metric over *X*, then the ordered pair (X,d) is termed as a 2-metric space.

Definition 1.2

A sequence $\{x_n\}$ in (X,d) is said to be a Cauchy sequence if for each $a\in X$, $\lim_{n,m\to\infty} d(x_n,x_m,a)=0$.

Definition 1.3

A sequence $\{x_n\}$ in (X,d) is said to be convergent to an element $x_0\in X$ if for each $a\in X$, $\lim_{n\to\infty} d(x_n,x_0,a)=0$. We denote it by $x_n\to x_0$ as $n\to\infty$.

In general every Cauchy sequence is not convergent in a 2-metric space. For this we refer to [10].

Definition 1.4

(*X*, *d*) is said to be a complete 2-metric space if every Cauchy sequence in (X,d) is convergent to an element of X.

Definition 1.5

A mapping $T:(X,d)\to(X,d)$ is said to be continuous at $x_0\in X$ if for each $a\in X$, $d(x_n,x_0,a)\to 0$ as $n\to\infty$ implies $d(Tx_n,Tx_0,a)\to 0$ as $n\to\infty$, equivalently $x_n\to x_0$ as $n\to\infty$ implies $Tx_n\to Tx_0$ as $n\to\infty$.

Definition 1.6

A mapping $G:X\to R$ is said to be *T* orbitally semi-continuous at $z\in X$ if $\{x_n\}$ is a sequence in $0(x,\infty)$ and $x_n\to z$ as $n\to\infty$ implies $G(z)\le\liminf_{n\to\infty}G(x_n)$.

Let $\phi:[0,\infty)\to[0,\infty)$ be a function such that

(i) ϕ is continuous and non decreasing

(ii) $\phi(t)=0$ if and only if $t = 0$.

Let us denote the collection of all such function ϕ by Φ.

We now denote by Ψ, the set of all functions $\psi:[0,\infty)\to[0,\infty)$ such that

(a) ψ is right continuous and non decreasing

(b) $\psi(t)<t$ for all $t>0$.

Definition 1.7

A map $T:(X,d)\to(X,d)$ is said to be (ϕ,ψ) contractive map if for each $a\in X$, $\phi(d(Tx,Ty,a))\le\psi(\phi(d(x,y,a)))$ for all $x,y\in X$.

Lemma 1.8

[4] If $\psi\in\Psi$ then $\lim_{n\to\infty}\psi^n(t)=0$ for each $t>0$ and if $\phi\in\Phi,\{a_n\}\subseteq[0,\infty)$ such that $\lim_{n\to\infty}\phi(a_n)=0$ then $\lim_{n\to\infty}a_n=0$.

Theorem 1.9

Let (*X*, *d*) be a complete 2-metric space and let $T:X\to X$ be a (ϕ,ψ) contractive map. Then there exist an unique $u\in X$ such that $u=Tu$.

Proof: Fix $x\in X$. Set $x_{n+1}=Tx_n$ with $x_0=x$. Then for each $a\in X$

$$\phi(d(x_n,x_{n+1},a))=\phi(d(Tx_{n-1},Tx_n,a))=\phi(d(Tx_{n-1},T^2x_{n-1},a))$$

$$\le\psi(\phi(d(x_{n-1},x_n,a)))\le\psi^2(\phi(d(x_{n-2},x_{n-1},a)))\le\ldots\le\psi^n(\phi(d(x_0,x_1,a)))$$

So by Lemma 1.8, $\lim_{n\to\infty}d(x_{n+1},x_n,a)=0$.

Now by the principle of mathematical induction we have $\lim_{n\to\infty} d\left(x_n, x_{n+k}, a\right) = 0, k = 1, 2, ...$

So $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exist a $u \in X$ such that $\lim_{n\to\infty} x_n = u \in X$, i.e. $\lim_{n\to\infty} d\left(x_n, u, a\right) = 0$ for each $a \in X$.

Now for each $a \in X$,

$$d\left(u, Tu, a\right) \le d\left(u, x_n, a\right) + d\left(x_n, Tu, a\right) + d\left(u, x_n, Tu\right)$$

and

$$\phi(d\left(x_n, Tu, a\right)) = \phi(d\left(Tx_{n-1}, Tu, a\right)) \le \psi(\phi(d\left(x_{n-1}, u, a\right))) \to 0 \text{ as } n \to \infty$$

Hence $\lim_{n\to\infty} \phi(d\left(x_n, Tu, a\right)) = 0$. So $d\left(u, Tu, a\right) = 0$ implying that $u = Tu$. Uniqueness of u is also clear.

Corollary 1.10

Let (*X*, *d*) be a 2-metric space. Let $\phi \in \Phi$ and $\psi \in \Psi$. Suppose that *T* is a continuous mapping of X onto itself such that *T* is $\left(\phi, \psi\right)$ contractive. If for some $x_0 \in X$ $x_n = T^n x_0 \to y \in X$ as $n \to \infty$ then y is the unique fixed point of T.

Proof: Clearly for each $a \in X, d\left(x_n, y, a\right) \to 0$ as $n \to \infty$. By routine calculation we get $\phi(d\left(x_n, x_{n+1}, a\right)) \le \psi^n(\phi(d\left(x_0, x_1, a\right)))$. So by Lemma 1.8, $d\left(x_n, Tx_n, a\right) \to 0$ as $n \to \infty$.

Now $d\left(Tx_{n,}, y, a\right) \le d\left(x_n, Tx_n, a\right) + d\left(x_n, y, a\right) + d\left(x_n, Tx_n, y\right) \to 0$ as $n \to \infty$ implying that $Tx_n \to y$ as $n \to \infty$. (1.10.1)

Clearly $\phi(d\left(Tx_n, T^2 x_n, a\right)) \le \psi(\phi(d\left(x_n, Tx_n, a\right))) \to 0$ as $n \to \infty$ and hence

$d\left(Tx_n, T^2 x_n, a\right) \to 0$ as $n \to \infty$.

Also $d\left(x_{n,}, T^2 x_n, a\right) \le d\left(x_n, Tx_n, a\right) + d\left(Tx_n, T^2 x_n, a\right) + d\left(x_n, Tx_n, T^2 x_n\right)$. (1.10.2)

But $\phi(d\left(x_n, Tx_n, T^2 x_n\right)) \le \psi(\phi(d\left(x_n, Tx_n, Tx_{n-1}\right))) = 0$. Consequently $d\left(x_n, Tx_n, T^2 x_n\right) = 0$

From (1.10.2) we have

$$d\left(x_{n,}, T^2 x_n, a\right) \le d\left(x_n, Tx_n, a\right) + d\left(Tx_n, T^2 x_n, a\right) \to 0 \text{ as } n \to \infty.$$

and so by (1.10.1), $T^2 x_n \to y$ as $n \to \infty$. Since *T* is continuous, we have

$$Ty = T\left(\lim_{n\to\infty} Tx_n\right) = \lim_{n\to\infty} T^2 x_n = y.$$ Uniqueness of y follows trivially.

Theorem 1.11

Let (*X*, *d*) be a complete 2-metric space suppose $T : X \to X$ be such that for each $a \in X$,

$\phi(d(Tx,Ty,a)) \le \psi(\phi(d(x,y,a))) \forall x,y \in X,$ where $\phi \in \Phi, \psi \in \Psi.$ Assume further that ϕ is sub-additive. Then

(i) $\lim_n T^n x = u$ exist

(ii) $\phi(d(T^n x,u,a)) \le \frac{\psi^n}{1-\psi}(\phi(d(x,Tx,a)))$ and

(iii) $d(u,Tu,a)=0$ if and only if $G(x)=d(x,Tx,a)$ is T-orbitally lower semi-continuous at $x \in X.$

Proof: By Theorem 1.9, (i) is clear. Let $x_0 \in X$ is fixed. Set $x_{n+1}=Tx_n$ with $x_0 = x$. Take for a positive integer n. Then for each $a \in X$

$\phi(d(T^n x_0, T^{n+1}x_0, a)) \le ... \le \psi^n(\phi(d(x_0,x_1,a))).$ Let *p* and *q* are integers such that $p > q$. Then

$$\phi(d(x_p,x_q,a)) \le \psi^q(\phi(d(x_0,x_{p-q},a)))$$

Also $d(x_{p-q},x_0,a) \le d(x_{p-q},x_{p-q-1},a) + d(x_{p-q},x_{p-q-1},x_0) + d(x_{p-q-1},x_0,a)$

Now $\phi(d(x_{p-q},x_{p-q-1},a)) \le \psi^{p-q-1}(\phi(d(x_0,x_1,a)))$

and so $\phi(d(x_{p-q},x_{p-q-1},x_0)) \le \psi^{p-q-1}(\phi(d(x_0,x_1,x_0))) \le \phi(d(x_0,x_1,x_0)) = 0$. So

$$d(x_{p-q},x_{p-q-1},x_0)=0.$$

Hence $d(x_{p-q},x_0,a) \le d(x_{p-q},x_{p-q-1},a) + d(x_{p-q-1},x_0,a)$

$$\le d(x_{p-q},x_{p-q-1},a) + d(x_{p-q-1},x_{p-q-2},a) + d(x_{p-q-2},x_0,a)$$
$$\le d(x_{p-q},x_{p-q-1},a) + d(x_{p-q-1},x_{p-q-2},a) + ... + d(x_1,x_0,a).$$

Using the property of $\phi,$ we get

$$\phi(d(x_{p-q},x_0,a)) \le \psi^{p-q-1}(\phi(d(x_1,x_0,a))) + \psi^{p-q-2}(\phi(d(x_1,x_0,a))) + ... + \phi(d(x_1,x_0,a))$$

Hence

$$\phi(d(x_p,x_q,a)) \le \psi^q\left[\psi^{p-q-1}(\phi(d(x_1,x_0,a))) + \psi^{p-q-2}(\phi(d(x_1,x_0,a))) + ... + \phi(d(x_1,x_0,a))\right]$$
$$\le \frac{\psi^q}{1-\psi}(\phi(d(x_1,x_0,a)))$$

i.e., $\phi(d(T^p x_0, T^q x_0, a)) \le \frac{\psi^q}{1-\psi}(\phi(d(x_1,x_0,a)))$. Letting $p \to \infty$, we have

$\phi(d(T^q x_0, u, a)) \leq \frac{\psi^q}{1-\psi}(\phi(d(Tx_0, x_0, a)))$ and so (ii) is proved.

It is clear that $d(u,Tu,a)$ = 0 implying that $G(u)$ is -orbitally lower semicontinuous at

Now $x_n = T^n x \to u$ as $n \to \infty$. Since G is T-orbitally lower semi-continuous at $u \in X$ we have

$$0 \leq G(u) = \phi(d(u,Tu,a)) \leq \liminf_{n\to\infty} \phi(G(x_n)) \leq \liminf_{n\to\infty} \psi(\phi(G(x_{n-1}))) \leq \ldots \leq \liminf_{n\to\infty} \psi^n(\phi(G(x_0, Tx_0, a))) = 0.$$

So $d(u,Tu,a) = 0.$

Theorem 1.12

Let (X, d) be a 2-metric space suppose $T : X \to X$ satisfies

$$\phi(d(Tx,Ty,a)) \leq \psi(\phi(d(x,y,a))) \forall x, y \in X$$

where $\phi \in \Phi$, $\psi \in \Psi$. If $F(T^n) \neq \phi$, the set of all fixed points of T^n, then T has a unique fixed point.

Moreover $F(T^n) = F(T)$.

Proof: If $n = 1$, then it is trvial. Take $n > 1$.Let $x \in X$ such that $x \neq Tx$. Now for each $a \in X$, $\phi(d(Tx, T^2x, a)) \leq \psi(\phi(d(x,Tx,a))) \forall x \in X$. Let $x \in F(T^n)$.

Now $\quad \phi(d(x,Tx,a)) = \phi(d(T^n x, T^{n+1}x, a))$

$$\leq \psi(\phi(d(T^{n-1}x, TT^{n-1}x, a))) \leq \ldots \leq \psi^n(\phi(d(x,Tx,a))) \to 0 \text{ as } n \to \infty$$

Using the property of ϕ, we have $d(x,Tx,a) = 0 \Rightarrow x = Tx$. So $x \in F(T)$ and hence $F(T^n) = F(T)$.

Theorem 1.13

Let (X, d) be a 2-metric space suppose. Let $\phi \in \Phi$ and $\psi \in \Psi$. Suppose that $T : X \to X$ be an orbitally lower semi-continuous map such that for each $a \in X$,

$$\phi(d(Tx,Ty,a)) \leq \psi(\phi(d(x,y,a))), \forall x, y \in X.$$

Then there exist a unique fixed point of T in X.

Proof: Let $x_0 \in X$. Let $x_n = T^n x_0$.Let $x_n \to u$ as $n \to \infty$.

As T is orbitally lower semi continuous. We have $d(u,Tu,a) \leq \liminf_{n\to\infty} d(x_n, x_{n+1}, a)$.

Now $\phi(d(u,Tu,a)) \le \phi\left(\liminf_{n\to\infty} d(x_n,x_{n+1},a)\right) = \phi(0) = 0$. So $d(u,Tu,a)=0$ implying that u is the fixed point of T. Uniqueness of u is also clear.

We now cite an example (see Example 1.14) of an operator which is not a contraction but (ϕ,φ) contractive operator having a unique fixed point.

Example 1.14

Let $X=\left\{0,1,\frac{1}{2},\frac{1}{3},\ldots\right\}$. Define of $d: X\times X\times X\to \mathbb{R}^+$ by $d(x,y,z)=1$ if *x*, *y*, *z* are distinct and $\left\{\frac{1}{n},\frac{1}{n+1}\right\}_{n\ge 1} \subset \{x,y,z\}$ for some positive integer *n* and $d(x,y,z)=0$ otherwise. Then (X,d) is a complete 2-metric space [12].

Define $T: X\to X$ by

$$T\left(\frac{1}{n}\right)=\frac{1}{n+1}, n\ge 1$$

and $T(0)=0$.

Define $\phi[0,\infty)\to[0,\infty)$ by

$$\phi(t)=0,\ t\in[0,1)$$
$$= \log t, t\ge 1.$$

Clearly ϕ is continuous and non decreasing satisfying $\phi(t)=0$ iff *t* = 0. So $\phi\in\Phi$.
Also define $\psi:[0,\infty)\to[0,\infty)$ by

$$\psi(t)=\frac{4}{5}\sqrt{t},\ t\in[0,\infty). \text{ Also } \psi(t)<t. \text{ So } \psi\in\Psi$$

Now the following possible cases arise. For $x,y,a\in X$

Case I: $d(x,y,a)=0, d(Tx,Ty,a)=0$, Case II: $d(x,y,a)=1, d(Tx,Ty,a)=0$ Case III: $d(x,y,a)=1, d(Tx,Ty,a)=1$, Case IV: $d(x,y,a)=0, d(Tx,Ty,a)=1$.

For case I: $\phi\big(d(Tx,Ty,a)\big)=\phi(0)=0$ and $\psi(\phi(d(x,y,a)))=\psi(\phi(0))=\psi(0)=0$

For case II: $\phi\big(d(Tx,Ty,a)\big)=\phi(0)=0$ and $\psi(\phi(d(x,y,a)))=\psi(\phi(1))=\psi(\log 1)=\psi(0)=0$

For case III: $\phi(d(Tx,Ty,a))=\phi(1)=\log 1=0$ and $\psi(\phi(d(x,y,a)))=\psi(\phi(1))=0$

For case IV: $\phi(d(Tx,Ty,a))=\phi(1)=\log 1=0$ and $\psi(\phi(d(x,y,a)))=\psi(\phi(0))=\psi(0)=0$

Thus we see that $T: X \to X$ is (ϕ, φ) contractive. If $x = 1, y = \frac{1}{2}$, then $Tx = \frac{1}{2}, Ty = \frac{1}{3}$. Take $a = \frac{1}{4}$. Hence $d(Tx, Ty, a) = 1$ and $d(x, y, a) = 1$. So there does not exist any $0 < \alpha < 1$ such that $d(Tx, Ty, a) \le \alpha d(x, y, a)$ holds. Hence T is not a contraction operator. Also T has a unique fixed point 0.

We now put an example (see Example 1.15) in which (X, d) is not complete but the mapping $T: X \to X$ is (ϕ, ψ) contractive without having a fixed point in X.

Example 1.15

Let $X = \mathbb{R}$ be the set of real numbers with a 2-metric defined by $d(x, y, z) = 0$ if at least two of three points are equal and $d(x, y, z) = 2$, otherwise. Then (X, d) is a 2-metric space. But it is not complete. Let ϕ, ψ are the functions as defined in Example 1.14.

Define $T: X \to X$ by $T(x) = 1 + x, x \in X$. But T has no fixed point in X. Although T is (ϕ, ψ) contractive.

We shall now show an example (see Example 1.16) of an operator $T: X \to X$ which is not (ϕ, ψ) contractive, but X is complete.

Example 1.16

Let $X = \mathbb{R}^+ \mathrm{x} \mathbb{R}^+$, and let d be a 2-metric which expresses $d(x, y, u)$ as area of the Euclidean triangle with vertices $x = (x_1, x_2), y = (y_1, y_2)$ and $u = (u_1, u_2)$. Then (X, d) is a complete 2-metric space (refer to [16]). Let $T: X \to X$ be defined by

$T(x, y) = 3(x, y), ((x, y) \ne (0, 0))$ and $T(0, 0) = (1, 1)$. T has no fixed point in X. Clearly T is not (ϕ, ψ) contractive.

2. SECTION-2

Definition 2.1

A mapping $T: (X, d) \to (X, d)$ is said to be a (ϕ, k) contractive mapping if for each $a \in X$ there exist a $0 < k < \frac{1}{2}$ such that

$$\phi(d(Tx, Ty, a)) \le k\left[\phi(d(x, Tx, a)) + \phi(d(y, Ty, a))\right] \text{ for all } x, y \in X.$$

Theorem 2.2

Let (X, d) be a complete 2-metric space. Let T be a (ϕ, k) mapping of X onto itself.

Then T has a unique fixed point X.. Also for each $x \in X$, the iterative sequence $\{T^n x\}_{n\geq 1}$ is converging to the fixed point of T.

Proof: Let $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n = T^n x_0$, for any $n \in \mathbb{N}$. Now for each $a \in X$,

$$\phi(d(Tx_0, T^2x_0, a)) \leq k\left[\phi(d(x_0, Tx_0, a)) + \phi(d(Tx_0, T^2x_0, a))\right]$$

and we get $\phi(d(Tx_0, T^2x_0 a)) \leq \left(\frac{k}{1-k}\right)\phi(d(x_0, Tx_0, a))$.

Now for each $a \in X, d(z, Tz, a) \leq d(z, T^n x_0, a) + d(T^n x_0, Tz, a) + d(z, T^n x_0, Tz)$ (2.2.1)

Also $\phi(d(T^n x_0, Tz, a)) \leq k\left[\phi(d(T^{n-1}x_0, T^n x_0, a)) + \phi(d(z, Tz, a))\right]$ (2.2.2)

and $\phi(d(T^{n-1}x_0, T^n x_0, a)) \leq k\left[\phi(d(T^{n-2}x_0, T^{n-1}x_0, a)) + \phi(d(T^{n-1}x_0, T^n x_0, a))\right]$. So

$$\phi(d(T^{n-1}x_0, T^n x_0, a)) \leq \left(\frac{k}{1-k}\right)\phi(d(T^{n-2}x_0, T^{n-1}x_0, a)) \leq \left(\frac{k}{1-k}\right)^2 \phi(d(T^{n-3}x_0, T^{n-2}x_0, a)) \leq \ldots \leq$$

$\leq \left(\frac{k}{1-k}\right)^n \phi(d(Tx_0, x_0, a))$. So $d(T^{n-1}x_0, T^n x_0, a) \to 0$ as $n \to \infty$.

Thus for each $a \in X$,

$\phi(d(x_n, x_{n+1}, a)) \leq \left(\frac{k}{1-k}\right)^n \phi(d(x_0, x_1, a))$ and hence by Lemma 1.8 $\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0$.

Thus for any positive integer k and for each $a \in X$ we get $\lim_{n\to\infty} d(x_n, x_{n+k}, a) = 0$. So $\{x_n\}$ is a Cauchy sequence in X and hence by completeness of X $T^n x_0 \to z$ as $n \to \infty$.
By using (2.2.1) and (2.2.2) we have for each $a \in X$, $d(z, Tz, a) \leq \lim_n d(T^n x_0, Tz, a)$ and so $\phi(d(z, Tz, a)) \leq \phi(\lim_n d(T^n x_0, Tz, a)) \leq \lim_n \phi(d(T^n x_0, Tz, a)) \leq k\phi(d(z, Tz, a))$. Hence $\phi(d(z, Tz, a)) = 0$ and so $d(z, Tz, a) = 0$, implying that $Tz = z$. Uniqueness of z is also clear.

Theorem 2.3

Let (X, d) be a 2-metric space. Let T be a (ϕ, k) contractive mapping of X onto itself. If for any positive integer n, $F(T^n) \neq \phi$ then T has a unique fixed point in X. Moreover $F(T^n) = F(T)$.

Proof: Let $T^n x = x(x \in X)$. For $n = 1$, the result is trivial. Put $x \neq Tx$. Now for each $a \in X$,

$$\phi(d(x,Tx,a)) = \phi(d(T^n x, T^{n+1}x, a)) \leq \left(\frac{k}{1-k}\right)^n \phi(d(x,Tx,a)) \to 0 \text{ as } n \to \infty.$$

So $d(x,Tx,a)=0$ and hence $x = Tx$. Clearly x is unique. The second part is trivial.

Theorem 2.4

Let (X,d) be a 2-metric space. Let T be a (ϕ,k) mapping of X onto itself.
Then there exist a unique fixed point in X.

Proof: Let $x_0 \in X$. Let $x_n = T^n x_0$. Let $x_n \to u \in X$ as $n \to \infty$. As *T* is continuous $T^n x_0 \to Tu$ as $n \to \infty$ and so $u = Tu$ and u is unique.

We now cite an example of a mapping which is (ϕ,k) contractive but not a contractive mapping in a 2-metric space. The following Example 2.5 supports our contention.

Example 2.5

Consider X and $T: X \to X$ as given in Example 1.14. Take $x = \frac{1}{2}, y = \frac{1}{3}$ and $a = \frac{1}{2}$ and we see that $d(Tx,Ty,a) < d(x,y,a)$ does not hold. Hence T is not contractive. Let $k = \frac{1}{3}$. Take $\phi(x) = 4x$. So $\phi \in \Phi$ .It is a routine exercise that T is a (ϕ,k) contractive mapping.

3. CONCLUSION

Existence of fixed points for a class of contractive mappings in a 2-metric space has been established by weakening the hypothesis, retaining the convergence property of successive iterates of maps as available in current literatures. Suitable examples have been cited either to support our theorems or to examine the validity of the conditions assumed in theorems.

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COMPETING INTERESTS

Authors have declared that they have no competing interests.

REFERENCES

1. Gähler S, 2-metric Raüme and ihre topologische structure. Math. Nachr. 1963;26:115-148.
2. Gähler S, Lineare 2-normierte Raüme. Math. Nachr. 1965;28:1-43.
3. Gähler S, Uber die uniformisierbarkeit 2-metriischer Raüme. Math. Nachr. 1965;28: 235-244.
4. Iseki K, Sharma PL, Sharma BK, Contraction type mapping on 2-metric space. Math. Japon. 1976;21(1):67-70.
5. Iseki K, Fixed point theorems in 2-metric spaces, Math. Seminar Notes, Kobe Univ. 1975;3:133-136.
6. Rhoades BE, Contractive type mappings on a 2-metric space. Math. Nachr. 1979; 91:151-155.
7. Saha M, Mukherjee RN. Fixed point of mappings with contractive iterates in a 2-metric space. Bull. Pure. Appl. Math. 2007;1(1):79-84.
8. Saha M, Dey D. On the theory of fixed points of contractive type mappings in a 2-metric space. Int. J. Math. Anal. 2009; 3(6):283-293.
9. Saha M, Dey D. Some results of fixed point of mappings in a 2-metric space. Int. J. Contemp. Math. Sci. 2009;4(21):1021-1028.

10. Saha M, Gangopadhyay M, Baisnab AP, Cantor's theorem in 2-metric space and its applications. Int. J. Math. Sci. and Engg. Appls. 2010;4(3):107-116.
11. Saha M, Dey D, Ganguly A. A generalization of fixed point theorems in a 2-metric space, Gen. Math. 2011;19(1): 87-98.
12. Naidu SVR, Prasad JR. Fixed point theorems in 2-metric spaces. Indian J. Pure. Appl. Math. 1986;17:974-993.
13. Boyd D, Wong JS. On nonlinear contraction. Proc. Amer. Math. Soc. 1969;20:456-464.
14. Bannch S. Surl es operations dans les ensembles abstraits et leur application aux equations untegrales, Fund. Math. 1922;3:133-181.
15. Kannan R. Some results on fixed points. Bull. Cal. Math Soc. 1968;60:71-76.
16. Liu Z, Zhang F, Mao J. Common fixed points for compatible mappings of type (A). Bull. Malyasian Math. Soc. 1999;22(2):67-86.

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