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Bayesian Inference Based on the Interval Grouped Data from the Weibull Model with Application

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Abstract

In this paper, based on the interval grouped data, Bayesian approach is used to obtain estimator for the Weibull scale parameter and some lifetime such as the reliability and hazard functions. The estimation procedures have been developed and modified under squared error loss function (SELF) and general entropy loss function (GELF). The estimators are derived using the inverted gamma conjugate prior. Credible intervals and high posterior density (HPD) credible intervals are also obtained. Prediction for the future number of failures in the corresponding intervals is presented. Finally, real life example is applied to illustrate the performance of the estimation procedures.

Keywords: *Weibull distribution; interval grouped data; bayesian estimation; loss functions; high posterior density estimator; credible intervals; Prediction.*

1 Introduction

The Weibull distribution has been extensively used in life testing and reliability studies. Initially, it was proposed by Weibull [1] for describing the fatigue failures from the wear out materials. Lieblein and Zelen [2] have used the model for the ball bearing failures. Recently, the Weibull statistics have become the most popular to describe the breakdown events in solid dielectrics [3]. Weibull distribution also provides a good model for lives of electrical and mechanical components or systems [4]. Reliability and survival analysis based on the weibull distribution are applied and discussed by many authors [5-9].

The density function for the Weibull distribution with parameters α, β is given by

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$$f(t) = \frac{\beta}{\theta} t^{\beta-1} \exp(-\frac{t^{\beta}}{\theta}) \quad ; \; t > 0 \; ; \theta > 0 \; ; \beta > 0 \tag{1}$$

Hence, the hazard rate and the reliability functions are given respectively by

$$h(t) = \frac{\beta}{\theta} t^{\beta-1} \quad , t > 0 \tag{2}$$

$$R(t) = \exp(\frac{-t^{\beta}}{\theta}) \quad , t > 0 \tag{3}$$

In the normal used conditions, the inspection units have long time for failure. Thus, having a complete failure data set is costly and time consuming. For this reason, censored life testing experiments are performed either by terminating the life test at a pre-assigned fixed single time, say T^* to have "Type I" censored data or after a pre-assigned number of failures at the random variable time $t(r)$ to have "Type II" censored data. Using both types of censoring, progressively censored data may be obtained when at different stages of the experiment; some of the surviving units remaining are censored from further observation. Examples are doubly, progressive type I and type II and hybrid progressive type I and type II censored data.

In other situations when continuous monitoring is not feasible, the intermittent inspection life testing is the only probable procedure by testing the failure units periodically. In the reliability and survival data analysis, the data obtained from the intermittent inspection are often called interval grouped data. Hence, the sample space of the life testing can be partitioned in different ways of selecting the end points to achieve the experimenter intention, the range and the source from which the data are collected. Interval grouped data are widely used in survival and reliability analysis [10-13].

Using different types of failure time's data, many researches are devoted for estimating the Weibull parameters. Nicholas et al. [14] has proved the uniqueness of the maximum likelihood estimators obtained from a complete data. He also presented a regression method approach using both complete and censored data. Using the censored data, maximum likelihood and least square regression estimators are applied to the Weibull parameters [15]. Nadarajah [16] has used progressively "Type II" censored data for the moments of the Weibull distribution. Kundu [17] presents maximum likelihood and Bayesian inference procedures for the parameters when the data are mixture of "Type I" and "Type II" censoring schemes. Other referees of interest may be found in Varian [18], Berger and Sun [19], Thompson and Basu [20], Zhang et al. [21], Chen et al. [22].

In order to select a single value δ as the Bayesian estimator of θ , a loss function must be specified. A frequently used symmetric loss function is the squared error loss function (SELF) defined by

$$L_s(\theta;\delta) = (\theta - \delta)^2 \tag{4}$$

Hence, the corresponding Bayesian estimator is the posterior mean [23].

In many practical situations, it appears more realistic to express the loss function in terms of the ratio $\frac{\delta}{\theta}$. In this case, a useful symmetric loss function is the general entropy loss function (GELF) proposed by Calabria and Pulcini [24] defined by

$$L_g(\theta;\delta) = (\frac{\delta}{\theta})^m - \theta \log(\frac{\delta}{\theta}) - 1 \tag{5}$$

Whose minimum occurs at $\delta = \theta$. This loss function is a generalization of the entropy loss which used by many authors [25-26]. Hence, the Bayesian estimator under GELF is obtained by

$$\theta_{Bg} = \left(E_{\tilde{\Pi}}\ (\theta^{-m})\right)^{\frac{-1}{m}} \tag{6}$$

where $\tilde{\Pi}$ is the corresponding posterior.

2. Likelihood, Prior and Posterior

Suppose that the lifetime T follow the Weibull distribution (θ, β) and assume the time scale line is divided into intervals by the constant inspection times a_j, $j = 1,2,..,k$. Assuming $a_0 = 0$ and $a_{k+1} = \infty$, then the corresponding intervals are expressed as $I_1 = [0,\ a_1)$, $I_2 = [a_1,\ a_2)$,... , $I_k = [a_{k-1},\ a_k), I_{k+1} = [a_k,\ \infty)$. If n units are put on the test for failure and the experimenter intermittently records the failure data which consists of the number of failures f_j in the intervals $I_j, j = 1,2...,k+1$ then the probability of failure in the corresponding intervals are

$$\begin{aligned} P_j(\theta,\beta) = R(a_{j-1}) - R(a_j) \quad &= \exp(\frac{-a_{j-1}^{\ \beta}}{\theta}) - \exp(\frac{-a_j^{\ \beta}}{\theta}) \\ &= \exp(\frac{-a_{j-1}^{\ \beta}}{\theta})\left(1 - \exp(\frac{a_j^{\ \beta} - a_{j-1}^{\ \beta}}{\theta})\right) \qquad , j = 1,2,..., k+1 \end{aligned} \tag{7}$$

Since $\sum_{j=1}^{k+1} f_j = n$, $\sum_{j=1}^{k+1} P_j(\theta) = 1$, the likelihood function given the interval grouped data is

$$L(\theta \mid \beta, G) \propto \exp(\frac{-\sum_{j=1}^{k+1} a_{j-1}^{\beta}\ f_j}{\theta}) \prod_{j=1}^{k} \left(1 - \exp(\frac{-(a_j^{\ \beta} - a_{j-1}^{\ \beta})}{\theta})\right)^{f_j} \tag{8}$$

A fundamental element in the Bayesian estimation is to specify a respective prior for the indexed parameter. In this research, we adopt the natural conjugate prior density for the Weibull scale parameter, namely the inverted gamma prior given by

$$\Pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{-(a+1)}\exp(\frac{-b}{\theta}) \qquad , a > 0,\ b > 0 \tag{9}$$

The mean and the variance of the above prior are

$$E(\theta) = \frac{b}{a-1} \qquad , \quad b > 0 \ \ , a > 1 \tag{10}$$

$$V(\theta) = b^2\left(\frac{1}{(a-1)^2(a-2)}\right) \quad , \ b > 0 \ , \ a > 2 \tag{11}$$

if $a = 0, b = 0$, then we get the non-informative prior.

Setting $S = \sum_{j=1}^{k+1} a_{j-1}^{\beta} \ f_j + b$ and combining (8) and (9), the posterior function of θ is

$$\Pi(\theta \mid G) = \frac{\theta^{-(a+1)}\exp(\frac{-S}{\theta})\prod_{j=1}^{k}(1-\exp(\frac{-(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{\theta}))^{f_j}}{\int_0^{\infty}\theta^{-(a+1)}\exp(\frac{-S}{\theta})\prod_{j=1}^{k}(1-\exp(\frac{-(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{\theta}))^{f_j}d\theta} \tag{12}$$

To obtain the integral in the denominator of (12), $\prod_{j=1}^{k}\left(1-\exp(\frac{-(a_j{}^{\beta}-a_{j-1}{}^{\beta})}{\theta})\right)^{fj}$ is approximated by substituting the average $Z = \frac{1}{k}\sum_{j=1}^{k}(a_j{}^{\beta} - a_{j-1}{}^{\beta})$ in both numerator and denominator of (12) and hence, the best posterior approximation can be given (Appendix 1) by

$$\Pi(\theta \mid G) = \frac{\sum_{j=0}^{r}\binom{r}{j}.(-1)^j\theta^{-(a+1)}\exp(\frac{-(jZ+S)}{\theta})}{\sum_{j=0}^{r}\binom{r}{j}.(-1)^j\int_0^{\infty}\theta^{-(a+1)}\exp(\frac{-(jZ+S)}{\theta})d\theta} \tag{13}$$

Where, $r = \sum_{j=1}^{k} f_j = n - f_{k+1}$

Using the transformations:

$$\theta = \frac{\beta t^{\beta-1}}{h(t)} \quad ,h(t) > 0 \quad , \quad \theta = \frac{t^{\beta-1}}{R(t)(\log(R(t))^2} \quad ,0 < R(t) < 1 \tag{14}$$

The posterior functions for both the hazard and the reliability functions are readily derived.

3. Bayesian Estimation

In this section, based on the interval grouped data, the Bayesian and the HPD estimators for the parameter θ are obtained and modified. Under SELF, the Bayesian estimator is given by the mean of the posterior, define:

$$\Psi(r,h,w) = \sum_{j=0}^{r} \binom{r}{j} (-1^j) h^{-w} \tag{15}$$

and taking the mean of $\Pi(\theta \mid \beta, G)$, the Bayesian estimator of θ under SELF is given from the following formula.

$$\theta_{SB} = \frac{\Psi(r, jZ + S, a-1)}{(a-1)\Psi(r, jZ + S, a)} \tag{16}$$

and under GELF of shape parameter m, the Bayesian estimator

$$\theta_{GB} = \left(\frac{\Gamma(a+m)}{\Gamma(a)}\right)^{\frac{-1}{m}} \left(\frac{\Psi(r, jZ+S, a+m)}{\Psi(r, jZ+S, a)}\right)^{\frac{-1}{m}} \tag{17}$$

To base the estimation on the sufficient statistics for θ given the interval grouped data $S = \sum_{j=1}^{k+1} a_{j-1}^{\beta} \; f_j + b$, the posterior can be approximated by

$$\Pi(\theta \mid \beta, G) = \frac{\theta^{-(a+r+1)} \exp(\frac{-S}{\theta})}{\int_0^{\infty} \theta^{-(a+r+1)} \exp(\frac{-S}{\theta}) \, d\theta} \tag{18}$$

hence, Bayesian estimator under SELF is

$$\theta_{SB=} \frac{S}{a+r-1} \tag{19}$$

and the corresponding Bayesian estimator under GELF is

$$\theta_{GB} = S\left(\frac{\Gamma(a+r+m)}{\Gamma(a+r)}\right)^{\frac{-1}{m}} \tag{20}$$

Mathematically, since the estimators in (19) and (20) are in closed form, they are sufficient for computing and modifications.

If it is accepted for some specific loss function, the estimation can be based on the maximum likelihood principle. In the Bayesian inference, this leads to the mode of the posterior density or the HPD estimator. Since the posterior defined in (14) is unimodal, the HPD estimator of θ denoted by θ_{HPD}, can be obtained by solving the equation

$$\exp\left(\frac{-S}{\theta}\right)\theta^{-(a+r+1)}\left(\frac{S}{\theta^2} - \frac{(a+r+1)}{\theta}\right) = 0 \tag{21}$$

This implies, the HPD estimator is given by

$$\theta_{HPD} = \frac{S}{a+r+1} \tag{22}$$

using the transformations in (14), the Bayesian estimators for the hazard rate and the reliability functions under SELF at a given specified t are respectively

$$h_s(t) = \frac{\beta(a+r)\,t^{\beta-1}}{S}\ , \qquad R_s(t) = \left(\frac{S}{S+t^{\beta}}\right)^{a+r} \qquad , t > 0 \tag{23}$$

Also, the Bayesian estimators for the hazard rate, and the reliability functions under GELF at a given specified time t are respectively

$$h_G(t) = \beta t^{\beta-1} S^{\frac{r+m}{m}}\left(\frac{\Gamma(a+r-m)}{\Gamma(a+r)}\right)^{\frac{-1}{m}}, R_G(t) = \left(\frac{S}{S-mt^{\beta-1}}\right)^{\frac{-1}{m}} \tag{24}$$

4. Credible Intervals

Another common Bayesian inference to obtain intervals $[C_1, C_2]$ for the unknown parameter θ is probably to lie. Based on the posterior distribution, the interval $[C_1, C_2]$ is said to be a $(1-\alpha)\%$ credible interval for θ if

$$\int_{c_1}^{c_2} \Pi(\theta \mid G) d\theta = 1 - \alpha \tag{25}$$

In choosing a credible interval for θ , it is usually desirable to minimize its size subject to the condition (25) which requires

$$\Pi(C_1 \mid G) = \Pi(C_2 \mid G) \tag{26}$$

an interval $[C_1, C_2]$ which simultaneously satisfies (25) and (26) is called the $(1 - \alpha)\%$ HPD credible interval. Substituting for $\Pi(\theta \mid G)$ from (18) and using the integral transform

$$\int_{h}^{\infty} \theta^{-(a+r+1)} \exp(\frac{-S}{\theta}) d\theta = S^{-(a+r)} \int_{0}^{\frac{S}{h}} w^{(a+r)-1} \exp(-w) dw$$

Implies HPD credible intervals can be obtained by choosing C_1, C_2 that satisfied the following two equations, simultaneously

$$I_g(\frac{S}{C_1}, a+r) - I_g(\frac{S}{C_2}, a+r) = 1 - \alpha \tag{27}$$

$$(\frac{C_1}{C_2})^{-(a+r+1)} = \exp(S(\frac{1}{C_1} - \frac{1}{C_2})) \tag{28}$$

where, $I_g(x, y) = \frac{1}{\Gamma(y)} \int_0^x t^{y-1} \exp(-t)\, dt$ is the incomplete gamma function

Using the transformations in (14), credible intervals and HPD credible intervals for the hazard and the reliability functions at a pre-given time t can be derived.

5. Prediction

In the Bayesian frame work related to life testing experiments, prediction is mainly concern with the future order failure time $t(r)$. In this research, we approach the prediction for the future number of failures in a pre-given interval.

Assume a new sample of size **N** is put on the test for failure, if the corresponding intervals $I_1 = [0,\ a_1)$, $I_2 = [a_1,\ a_2)$,... , $I_k = [a_{k-1}, a_k), I_{k+1} = [a_k,\ \infty)$ are still fixed and the number of failures $f_j, j = 1,2,..., k+1$ in these intervals for the pre sample of size n are given, then the distribution for each of the new number of failures

$f_j^*, j = 1,2,...,k+1$ for a new sample of size N is binomial $(\mathbf{N}, P_j(\theta))$ where $P_j(\theta)$ is as defined in (7). This implies the predictive posterior of f_j^* which is given by

$$\Pi^*(f_j^* = x) = \frac{\binom{N}{x}\int_0^\infty \left(P_j(\theta)\right)^x (1-P_j(\theta))^{N-x}\, \theta^{-(a+r+1)} \exp(\frac{-S}{\theta})d\theta}{\int_0^\infty \theta^{-(a+r+1)} \exp(\frac{-S}{\theta})d\theta}, x = 0,1,2,...,m \quad (29)$$

In particular, the predictive posterior of f_1^*, f_{k+1}^* are given by substituting $P_1(\theta) = 1 - \exp(\frac{-a_1^\beta}{\theta})$, $P_{k+1}(\theta) = \exp(\frac{-a_k^\beta}{\theta})$ for $P_j(\theta)$ in (29), respectively.

This implies, the expected prediction value of f_1, f_{k+1} are

$$E_{\Pi^*}(f_1^*) = \frac{S^{a+r}}{\Gamma(a+r)}\sum_{x=0}^{N}\binom{N}{x}\frac{x\, q_1^x\, S^{a+r}\,\Gamma(a+r+x)}{(S+a_1^\beta\ (N-x))^{a+r+1}} \quad (30)$$

$$E_{\Pi^*}(f_{k+1}^*) = \frac{S^{a+r}}{\Gamma(a+r)}\sum_{x=0}^{N}\binom{N}{x}\frac{x\, q_{k+1}^x\ \ S^{a+r}\,\Gamma(a+r+x)}{(S+a_{k+1}^\beta(N-x))^{a+r+1}} \quad (31)$$

6. Application and Conclusion

To illustrate the performance of the Bayesian inference, real life example is presented in this section [27]. The data represent number of million revolutions before failure for 23 ball bearings dividing each by 17. The data are as follows:

1.05, 1.70, 1.94, 2.44, 2.48, 2.68, 2.85, 3.05, 3.06, 3.18, 3.27, 3.99, 4.04, 4.04, 4.05, 4.95, 5.48, 5.80, 6.18, 6.23, 7.52, 7.53, 10.20.

Since the interval grouped data have specific loss of information about the exact failure times, the obtained Bayesian estimator derived in this paper are compared with the maximum likelihood estimator using the complete ungrouped data. The above ungrouped data are fit to the Weibull distribution by the maximum likelihood estimation method using Minitab with 95% confidence interval. The corresponding maximum likelihood estimators using the complete ungrouped data for the scale and the shape parameters are $\hat{\alpha}=4.81615, \hat{\beta}=2.1018$ implies the value of θ, $\hat{\theta}_{mle} = 27.2228$. Then the data are grouped into intervals of fixed length = 1.5, This length is probable to the above data range to terminate the experiment at the times $a_k =$ 4.5, 6 and 7.5 to have 3,4,5 intervals. Two cases for the informative priors are considered. In the first case assuming $\text{var}(\theta) = 10$ and in the second case $\text{var}(\theta) = 5$. In both cases, the prior mean is assumed to satisfy $E_{\Pi}(\theta) = \hat{\theta}_{mle} = 27.2228$. Thus, the available prior information indicates **a=** 76.108, b= 2044.65 for the first case and a= 150.2161, b= 4062.08 for the second. For the general loss the value of m is fixed to be 2 in both cases. The Bayesian estimators θ_{SB}, θ_{GB} and θ_{HPD} are computed assuming the end interval points $a_k =$ 4.5, 6 and 7.5 using equations (19), (20) and (22) and their efficiencies is computed as the ratio of their values to $\hat{\theta}_{mle}$. 90% symmetric Credible Intervals are derived using (27) and (28). The predicted values for the number of failures are also computed using (30) and (31) as illustrated in Tables 1 and 2.

Table 1. Case 1 performance with a= 76.108, and b= 2044.65

a_j	θ_{SB} **efficiency**		θ_{GB} **efficiency**		θ_{HPD} **efficiency**		**90% Credible Interval**	**Real and Predicted value s of**	
								f_{k+1}	f^*_{k+1}
4.5	24.5267	90.09%	24.1254	88.62%	23.9941	88.14%	(20.60,29.40)	8	7.35
6	24.6645	90.60%	24.2738	89.16%	24.1459	88.69%	(20.75,29.33)	5	4.42
7.5	26.3243	96.69%	25.9160	95.19%	25.7822	94.71%	(22.55,29.21)	3	2.85

Table 2. Case 2 performance with a= 150.2161, and b= 4062.08

a_j	θ_{SB} **efficiency**		θ_{GB} **efficiency**		θ_{HPD} **efficiency**		**90% Credible Interval**	**Real and Predicted values of**	
								f_{k+1}	f^*_{k+1}
4.5	25.7434	94.55%	25.5105	93.71%	25.4336	93.42%	(22.62,29.24)	8	7.76
6	26.2696	96.49%	25.5691	93.92%	25.9535	95.33%	(22.70,29.23)	5	4.71
7.5	27.2000	99.91%	26.4832	97.28%	26.2312	96.35%	(23.53,29.20)	3	2.96

Despite there is a considerable loss of information about the exact failure times. From the results in Tables 1 and 2, Bayesian inference based on the interval grouped data show significance efficiency as compared to the maximum likelihood estimation using the complete ungrouped data. This efficiency increases as the number of intervals increases. This result is satisfied, as the number of interval increases, the loss of information about the exact failure times decreases. The available prior information clearly affects the accuracy of the Bayesian estimates. This depends on the variance of the indicated priors. The developed computations for both the posterior and the corresponding estimators and prediction gave a high precision and thus, can be used for any further inference.

Competing interests

Authors have declared that no competing interests exist.

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Appendix 1

Proof for the posterior approximation in equations (12) and (13)
Let;

$$w1 = min\{(a_j{}^{\beta} - a_{j-1}{}^{\beta}), j = 1,2, \dots, k\}$$
$$w2 = max\{(a_j{}^{\beta} - a_{j-1}{}^{\beta}), j = 1,2, \dots, k\}$$
$$Z = avarge\{(a_j{}^{\beta} - a_{j-1}{}^{\beta}), j = 1,2, \dots, k\} = \frac{\sum_{j=1}^{k}(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{k}$$

This implies,

$$1 - e^{-\frac{w2}{\theta}} \leq 1 - e^{-\frac{(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{\theta}} \leq 1 - e^{-\frac{Z}{\theta}} \leq 1 - e^{-\frac{w1}{\theta}}, j = 1,2, \dots, k$$

This implies,

$$\prod_{j=1}^{k}(1 - e^{-\frac{w2}{\theta}})^{f_j} \leq \prod_{j=1}^{k}\left(1 - e^{-\frac{(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{\theta}}\right)^{f_j} \leq \prod_{j=1}^{k}\left(1 - e^{-\frac{Z}{\theta}}\right)^{f_j}$$
$$\leq \prod_{\mathrm{j}=1}^{\mathrm{k}}(1 - e^{-\frac{w1}{\theta}})^{f_j}$$

Thus, substitute for $\prod_{j=1}^{k}\left(1 - e^{-\frac{(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{\theta}}\right)^{f_j}$

$\prod_{j=1}^{k}(1 - e^{-\frac{w2}{\theta}})^{f_j}$ in the denominator of (12)
$\prod_{\mathrm{j}=1}^{\mathrm{k}}(1 - e^{-\frac{w1}{\theta}})^{f_j}$ in the numerator of (12)

We will have a lower bound for the posterior in (12) which gives a lower bound estimate for the Bayesian estimator.

Similarly, substitute for $\prod_{j=1}^{k}\left(1 - e^{-\frac{(a_j{}^{\beta} - a_{j-1}{}^{\beta})}{\theta}}\right)^{f_j}$

$\prod_{j=1}^{k}(1 - e^{-\frac{w1}{\theta}})^{f_j}$ in the denominator of (12)
$\prod_{\mathrm{j}=1}^{\mathrm{k}}(1 - e^{-\frac{w2}{\theta}})^{f_j}$ in the numerator of (12)

Then, we will have an upper bound for the posterior which gives an upper bound estimate for the Bayesian estimator.

Therefore to have a best approximation estimate for the posterior and the Bayesian estimator, substitute for

$\prod_{j=1}^{k}\left(1-e^{-\frac{\left(a_j{}^{\beta}-a_{j-1}{}^{\beta}\right)}{\theta}}\right)^{f_j}$, $\prod_{j=1}^{k}\left(1-e^{-\frac{Z}{\theta}}\right)^{f_j}$ in both numerator and denominator of (12) and set $r = n - f_{k+1}$ and use the binomial expansion

$$\prod_{j=1}^{k}\left(1-e^{-\frac{Z}{\theta}}\right)^{f_j} = (1-e^{-Z})^{n-f_{k+1}} = \sum_{j=0}^{n-f_{k+1}}(-1)^j\binom{n-f_{k+1}}{j}e^{-jZ}$$

Then, simplify, you will have exactly equation (13).

Note: $e^x = \exp(x)$

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