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Some Strongly Summable Double Sequence Spaces over n-normed Spaces

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Abstract

In this paper we introduce some strongly summable double sequence spaces defined by a sequence of Orlicz functions over n-normed spaces. We also study some topological properties and inclusion relation between these spaces.

Keywords: Paranorm space; Orlicz function; solid, strongly summable sequences; double sequences; n-normed space.

1 INTRODUCTION

A double sequence is a double infinite array of elements $x_{k,l} \in \mathbb{R}$ for all $k, l \in \mathbb{N}$. The initial works on double sequences is found in (Bromwich, 1965). Later on, it was studied by Hardy (1917), Moricz (1991), Moricz and Rhoades (1988), Tripathy (2003, 2004), Basarir and Sonalcan, (1999) and many others. Hardy (1917) introduced the notion of regular convergence for double sequences. The concept of paranormed sequences was studied by (Nakano, 1951) and (Simmons, 1951) at the initial stage. The concept of 2-normed spaces was initially developed by (Gahler, 1961) in the mid of 1960's while that of n-normed spaces one can see in (Misiak 1999). Since, then many others have studied this concept and obtained various results (Gunawan , 2001; Gunawan and Mashadi, 2001). By the convergence of a double sequence we mean the convergence of the Pringsheim sense i.e. a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by P – lim x = L) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$, (Pringsheim, 1900). We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. Let l''_∞ the space of all bounded double sequence such that $\| x_{k,l} \|_{\infty,2} = sup_{k,l}|x_{k,l}| < \infty$.

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The notion of difference sequence spaces was introduced by (Kizmaz, 1981), who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak (1995) by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.
Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces $Z(\Delta_s^m) = \{ x = (x_k) \in w : (\Delta_s^m x_k) \epsilon Z\}$,

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \epsilon \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+sv} .$$

Taking s=1, we get the spaces which were studied by Et and Colak (1995). Taking m = s = 1, we get the spaces which were introduced and studied by Kizmaz (1981).

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri, (1971) used the idea of Orlicz function to define the following sequence space,

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\},$$

which is called as an Orlicz sequence space. The space l_M is a Banach space with the norm

$$\| x \| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

It is shown in Lindenstrauss and Tzafriri, (1971) that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \ge 1)$. The Δ_2-condition is equivalent to $M(Lx) \le k\, LM(x)$ for all values of $x \ge 0$, and for $L > 1$.

A double sequence space E is said to be solid if $\alpha_{k,l} x_{k,l} \epsilon E$ whenever $x_{k,l} \epsilon E$ and for all double sequences $\alpha_{k,l}$ of scalars with $|\alpha_{k,l}| \le 1$, for all $k, l \epsilon \mathbb{N}$.

A sequence space E is said to be monotone if it contains the canonical preimages of all its step spaces.

Let $\lambda = (\lambda_r)$ be a non-decreasing sequence of positive numbers tending to infinity and $\lambda_{r+1} \le \lambda_r + 1, \lambda_1 = 1$. The generalized de la Vallee-Poussin mean is defined by

$$t_r(x) = \frac{1}{\lambda_r} \sum_{k \epsilon I_r} x_k,\ I_r = [r - \lambda_r + 1, r].$$

A single sequence $x = (x_k)$ is said to be (V,λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ (Leinder, 1965). If $\lambda_r = r$, then the (V,λ)-summability is reduced to $(C,1)$-summability (Silverman, 1913; Toeplitz, 1913).

The double sequence $\lambda_2 = (\lambda_{m,n})$ of positive real numbers tending to infinity such that

$$\lambda_{m+1,n} \leq \lambda_{m,n} + 1, \lambda_{m,n+1} \leq \lambda_{m,n} + 1,$$
$$\lambda_{m,n} - \lambda_{m+1,n} \leq \lambda_{m,n+1} - \lambda_{m+1,n+1}, \lambda_{1,1} = 1,$$

and

$$I_{m,n} = \{(k,l): m - \lambda_{m,n} + 1 \leq k \leq m, n - \lambda_{m,n} + 1 \leq l \leq n\}.$$

The generalized double de la Vallee-Poussin mean is defined by

$$t_{m,n} = t_{m,n}(x_{k,l}) = \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} x_{k,l}.$$

A double number sequence $x = (x_{k,l})$ is said to be (V_2,λ_2)-summable to a number L if $P - \lim_{m,n} t_{m,n} = L$. If $\lambda_{m,n} = mn$, then the (V_2,λ_2)-summability is reduced to $(C,1,1)$-summability see (Robinson, 1926). We write

$$[V_2,\lambda_2] = \left\{ x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} |x_{k,l} - L| = 0, for\ some\ L \right\}$$

for set of double sequence $x = (x_{k,l})$. We say that $x = (x_{k,l})$ is strongly $[V_2,\lambda_2]$-summable to L, that is $x = (x_{k,l}) \to L$ $([V_2,\lambda_2])$.

Let $n \in \mathbb{N}$ and X be a linear space over the field K, where K is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|.,\ldots,.\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;
(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;
(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in K$, and
(4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n-norm on X and the pair $(X, \|.,\ldots,.\|)$ is called a n-normed space over the field K. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $\|x_1, x_2, \ldots, x_n\|_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$. Let $(X, \|.,\ldots,.\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be linearly dependent set in X. Then the following function $\|.,\ldots,.\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \dots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in a normed space $(X, \|., \dots, .\|)$ is said to converge to some $L \epsilon X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \; for\ every\ z_1, \dots, z_{n-1} \epsilon X.$$

A sequence (x_k) in a normed space $(X, \|., \dots, .\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \; for\ every\ z_1, \dots, z_{n-1} \epsilon X.$$

If every Cauchy sequence X converges to some $L \epsilon X$, then $X$is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,
(2) $p(-x) = p(x)$, for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda_n$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormd space. It is well known that the metric of any linear metric space is given by some total paranorm (see (Wilansky, 1984, Theorem 10.4.2, p-183). For more details about sequence spaces please see Esi (2009, 2011), Raj et al. (2010, 2011) and Tripathy et al., (2005).

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a bounded sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} = H$ and let $K = \max\{1, 2^{H-1}\}$.
Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

(1.1) $$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K\left(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\right).$$

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers. In the present paper, we define the following sequence spaces:

$$[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_0 = \left\{x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0,\ for\ some\ \rho > 0\right\},$$

$$[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_1 = \left\{x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0,\ for\ some\ some\ L\ and\ \rho > 0\right\}$$

and

$$[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_\infty = \left\{x = (x_{k,l}) : \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty, \; for\ some\ \rho > 0\right\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$[V_2, \boldsymbol{\lambda_2}, \Delta, p, \|., \dots, .\|]_0 = \left\{x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0, \; for\ some\ \rho > 0\right\},$$

$$[V_2, \boldsymbol{\lambda_2}, \Delta, p, \|., \dots, .\|]_1 = \left\{x = (x_{k,l}) : P - \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[\left(\left\|\frac{\Delta x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0, \; for\ some\ some\ L\ and\ \rho > 0\right\}$$

and

$$[V_2, \boldsymbol{\lambda_2}, \Delta, p, \|., \dots, .\|]_\infty = \left\{x = (x_{k,l}) : \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty, \; for\ some\ \rho > 0\right\}.$$

If we take $p = (p_{k,l}) = 1$, we get

$$[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, \|., \dots, .\|]_0 = \left\{x = (x_{k,l}) : \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right] = 0, \; for\ some\ \rho > 0\right\},$$

$$[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, \|., \dots, .\|]_1 = \left\{x = (x_{k,l}) : \lim_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l} - L}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right] = 0, \; for\ some\ L\ and\ \rho > 0\right\}$$

and

$$[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, \|., \dots, .\|]_\infty = \left\{x = (x_{k,l}) : \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right] < \infty, \; for\ some\ \rho > 0\right\}.$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces.

2 SOME TOPOLOGICAL PROPERTIES

In this section of the paper we study very interesting properties like linearity, paranorm and some attractive inclusion relation between the spaces $[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_0$, $[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_1$ and $[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_\infty$.

Theorem 2.1 Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be any bounded factorable double sequence of positive real numbers. Then the spaces

$[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_0, [V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_1$ *and* $[V_2, \boldsymbol{\lambda_2}, \boldsymbol{\mathcal{M}}, \Delta, p, \|., \dots, .\|]_\infty$ are linear spaces over the field of complex number $\mathbb{C}$.

Proof. Let $x = (x_{k,l}),\ y = (y_{k,l}) \in [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$. Let $\alpha, \beta \in \mathbb{C}$. Then there exist positive real number ρ_1, ρ_2 such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0$$

and

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} = 0.$$

Let $\rho_{3} = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M_{k,l}$'s are non-decreasing and convex and therefore by using inequality (1.1), we have

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta(\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta \alpha x_{k,l}}{\rho_3}, z_1, \dots, z_{n-1}\right\|\right) + \left(\left\|\frac{\Delta\, \beta y_{k,l}}{\rho_3}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \frac{1}{2^{p_{k,l}}}\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$+ K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \frac{1}{2^{p_{k,l}}}\left[M_{k,l}\left(\left\|\frac{\Delta y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$

$$\leq K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$+ K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\in I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$< \infty.$$

Thus $\alpha x + \beta y \in [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$. This proves that $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$ is a linear space. Similarly, we can prove that $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_0$ and $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_1$ are linear spaces.

Theorem 2.2 Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and $p = (p_{k,l})$ be any bounded factorable double sequence of positive real numbers, then the space $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots \dots, .\|]_\infty$ is a paranormed space, paranormed by

$$g(x) = \inf\left\{(\rho)^{\frac{p_{k,l}}{H}} : \left(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1\right\},$$
$$where\ 0 < p_{k,l} \leq \sup p_{k,l} = G, H = \max(1, G).$$

Proof. (i) Clearly $g(x) \geq 0$ for $x = (x_{k,l}) \epsilon [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$. Since $M_{k,l}(0) = 0$, we get $g(0) = 0$.

(ii) $g(-x) = g(x)$

(iii) Let $x = (x_{k,l}), y = (y_{k,l}) \epsilon [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$, then there exist $\rho_1, \rho_2 > 0$, such that

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \leq 1,$$

and

$$\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by using Minkowski's inequality, we have

$$\left[M_{k,l}\left(\left\|\frac{\Delta(x_{k,l} + y_{k,l})}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$= \left[M_{k,l}\left(\left\|\frac{\Delta(x_{k,l} + y_{k,l})}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1}\right\|\right) + \left(\left\|\frac{\Delta y_{k,l}}{\rho_1 + \rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right) + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right)\left(\left\|\frac{\Delta y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$

and thus

$$g(x, y) = \inf\left\{(\rho_1 + \rho_2)^{\frac{p_{k,l}}{H}} : \left(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta(x_{k,l} + y_{k,l})}{(\rho_1 + \rho_2)}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1\right\}$$
$$\leq \inf\left\{(\rho_1)^{\frac{p_{k,l}}{H}} : \left(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \leq 1\right\}$$

$$+ \inf\left\{(\rho_2)^{\frac{p_{k,l}}{H}} : \left(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta y_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \le 1\right\}.$$

Now, let $\lambda \epsilon \mathbb{C}$, then the continuity of the product follows from the following inequality:

$$g(\lambda x) = \inf\left\{(\rho)^{\frac{p_{k,l}}{H}} : \left(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta \lambda x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \le 1\right\}$$

$$= \inf\left\{(|\lambda| s)^{\frac{p_{k,l}}{H}} : \left(\frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{s}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}\right)^{\frac{1}{H}} \le 1\right\},$$

where $s = \frac{\rho}{|\lambda|}$. This completes the proof of the theorem.

Theorem 2.3 If $0 < p_{k,l} < q_{k,l} < \infty$ *for each* k *and* l, then

$[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty \subseteq [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, q, \|., \dots, .\|]_\infty$.

Proof. Let $x = (x_{k,l}) \epsilon [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$, then there exists some $\rho > 0$ such that

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty.$$

This implies that

$$M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)^{p_{k,l}} < 1,$$

for sufficiently large value of k and l. Since $M_{k,l}$'s are non-decreasing, we get

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{q_{k,l}}$$
$$\le \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$< \infty.$$

Thus, $x = (x_{k,l}) \epsilon [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, q, \|., \dots, .\|]_\infty$. This completes the proof of the theorem.

Theorem 2.4 (i) If $0 < \inf p_{k,l} \le p_{k,l} < 1$, then

$$[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty \subset [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, \|., \dots, .\|]_\infty.$$

(ii) If $1 \le p_{k,l} \le \sup p_{k,l} < \infty$, then

$$[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, \|., \dots, .\|]_\infty \subset [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty.$$

Proof. (i) Let $x = (x_{k,l}) \epsilon [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$. Since $0 < \inf p_{k,l} \le 1$, we have

$$\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\epsilon\, I_{m,n}}\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]$$
$$\leq \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\epsilon\, I_{m,n}}\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}},$$

and hence $x=(x_{k,l})\epsilon\,[V_2,\boldsymbol{\lambda_2},\mathcal{M},\Delta,\|.,\ldots,.\|]_\infty$.

(ii) Let $p_{k,l}$ for each (k,l) and $\sup_{k,l}p_{k,l}<\infty$. Let $x=(x_{k,l})\epsilon\,[V_2,\boldsymbol{\lambda_2},\mathcal{M},\Delta,\|.,\ldots,.\|]_\infty$.

Then, for each $0<\varepsilon<1$, there exists a positive integer $\mathbb{N}$ such that

$$\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\epsilon\, I_{m,n}}\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]\leq\varepsilon<1,$$

for all $m,n\geq N$. This implies that

$$\sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\epsilon\, I_{m,n}}\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right]^{p_{k,l}}$$
$$\leq \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\epsilon\, I_{m,n}}\left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho},z_1,\ldots,z_{n-1}\right\|\right)\right].$$

Thus $x=(x_{k,l})\epsilon\,[V_2,\boldsymbol{\lambda_2},\mathcal{M},\Delta,p,\|.,\ldots,.\|]_\infty$ and this completes the proof.

Theorem 2.5 For any sequence of Orlicz functions $\mathcal{M}=(M_{k,l})$ which satisfies Δ_2-condition, we have

$$[V_2,\lambda_2,\Delta,p,\|.,\ldots,.\|]\subset[V_2,\lambda_2,\mathcal{M},\Delta,\|.,\ldots,.\|].$$

Proof. Let $x=(x_{k,l})\epsilon\;[V_2,\lambda_2,\Delta,p,\|.,\ldots,.\|]$, then
$A_{m,n}=P-\ \sup_{m,n}\frac{1}{\lambda_{m,n}}\sum_{(k,l)\epsilon\, I_{m,n}}\left|\Delta x_{k,l}-L\right|^{p_{k,l}}$ for some L.
Let $\varepsilon>0$ and choose δ with $0<\delta<1$ such that $M_{k,l}(t)<\varepsilon$ for $0\leq t\leq\delta$. Write $y_{k,l}=\left|\Delta x_{k,l}-L\right|$ and consider

$$\sum_{(k,l)\epsilon\, I_{m,n}}\left[M_{k,l}(y_{k,l})\right]^{p_{k,l}}=\sum_1+\sum_2,$$

where the first summation is over $y_{k,l}\leq\delta$ and the second summation over $y_{k,l}>\delta$. Since $\mathcal{M}=(M_{k,l})$ is continuous. $\sum_1<\varepsilon$ and for $y_{k,l}>\delta$, we use the fact that $y_{k,l}<\frac{y_{k,l}}{\delta}<1+\frac{y_{k,l}}{\delta}$.
Since $\mathcal{M}=(M_{k,l})$ is non-decreasing and convex, it follows that

$$M_{k,l}(y_{k,l})<M_{k,l}\left(1+\frac{y_{k,l}}{\delta}\right)<\frac{1}{2}M_{k,l}(2)+\frac{1}{2}M_{k,l}\left(2\frac{y_{k,l}}{\delta}\right).$$

Since $\mathcal{M}=(M_{k,l})$ satiesfies Δ_2-condition, therefore

$$M_{k,l}(y_{k,l})<\frac{1}{2}K\frac{y_{k,l}}{\delta}M_{k,l}(2)+\frac{1}{2}K\frac{y_{k,l}}{\delta}M_{k,l}(2)=K\frac{y_{k,l}}{\delta}M_{k,l}(2).$$

Hence,
$\sum_2 < \max\left(1, K\delta^{-1}M_{k,l}(2)\right)^H A_{m,n}$, where $H = \sup_{k,l} p_{k,l}$. This proves that
$$[V_2, \lambda_2, \Delta, p, \|., \dots, .\|] \subset [V_2, \lambda_2, \mathcal{M}, \Delta, \|., \dots, .\|].$$

Theorem 2.6 Let $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ are sequences of Orlicz functions, then we have

$$[V_2, \boldsymbol{\lambda_2}, \mathcal{M}', \Delta, p, \|., \dots, .\|]_\infty \cap [V_2, \boldsymbol{\lambda_2}, \mathcal{M}'', \Delta, p, \|., \dots, .\|]_\infty \subseteq [V_2, \boldsymbol{\lambda_2}, \mathcal{M}' + \mathcal{M}'', \Delta, p, \|., \dots, .\|]_\infty.$$

Proof. Let $x = (x_{k,l}) \epsilon\, [V_2, \boldsymbol{\lambda_2}, \mathcal{M}', \Delta, p, \|., \dots, .\|]_\infty \cap [V_2, \boldsymbol{\lambda_2}, \mathcal{M}'', \Delta, p, \|., \dots, .\|]_\infty.$
Then

$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M'_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_1}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty$, for some $\rho_1 > 0$
and
$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M''_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho_2}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty$, for some $\rho_2 > 0$.

Let $\rho = \max(\rho_1, \rho_2)$. The results follows from the inequality

$$\begin{aligned}
&\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[\left(M'_{k,l} + M''_{k,l}\right)\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&= \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M'_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right) + M''_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\le K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M'_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} \\
&\quad + K \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M''_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}}.
\end{aligned}$$

Theorem 2.7 For any sequence of Orlicz functions $\mathcal{M} = (M_{k,l})$ and $p = (p_{k,l})$ be bounded double sequence of strictly positive real numbers. Then (i) $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_0 \subset [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$.

(ii) $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_1 \subset [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$.

Proof. The proof is easy so we omit it.

Theorem 2.8 The sequence space $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$ is solid.

Proof. Let $x = (x_{k,l}) \epsilon\, [V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$, i.e
$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l)\epsilon\, I_{m,n}} \left[M_{k,l}\left(\left\|\frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1}\right\|\right)\right]^{p_{k,l}} < \infty.$$

Let $(\alpha_{k,l})$ be double sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N} \times \mathbb{N}$. Then, we get

$$\sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta \alpha_{k,l} x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}}$$
$$\leq \sup_{m,n} \frac{1}{\lambda_{m,n}} \sum_{(k,l) \in I_{m,n}} \left[M_{k,l} \left(\left\| \frac{\Delta x_{k,l}}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_{k,l}}$$

and this completes the proof.

Theorem 2.9 The sequence space $[V_2, \boldsymbol{\lambda_2}, \mathcal{M}, \Delta, p, \|., \dots, .\|]_\infty$ is monotone.

Proof. The proof follows from Theorem 2.8.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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