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Research Article

On the New Generalized Hahn Sequence Space h_d^p

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In this article, we define the new generalized Hahn sequence space h_d^p, where $d=(d_k)_{k=1}^\infty$ is monotonically increasing sequence with $d_k \neq 0$ for all $k \in \mathbb{N}$, and $1<p<\infty$. Then, we prove some topological properties and calculate the $\alpha-$, $\beta-$, and $\gamma-$duals of h_d^p. Furthermore, we characterize the new matrix classes (h_d, λ), where $\lambda=\{bv, bv_p, bv_\infty, bs, cs,\}$, and (μ, h_d), where $\mu=\{bv, bv_0, bs, cs_0, cs\}$. In the last section, we prove the necessary and sufficient conditions of the matrix transformations from h_d^p into $\lambda=\{\ell_\infty, c, c_0, \ell_1, h_d, bv, bs, cs\}$, and from $\mu=\{\ell_1, bv_0, bs, cs_0\}$ into h_d^p.

1. Introduction and Basic Notations

The set of all complex valued sequences is denoted by ω and each vector subspace of ω is called a sequence space. The sets ℓ_∞, c, and c_0 are bounded, convergent and null sequence spaces, respectively. Moreover, ℓ_p, ℓ_1, cs, cs_0, bs and bv are the spaces of all absolutely $p-$summable, absolutely summable, convergent series, null series, bounded series and seqeuences of bounded variation, respectively, where $1<p<\infty$.

The alpha-dual λ^α, beta-dual λ^β and gamma-dual λ^γ of a sequence space λ are defined by

$$\begin{aligned}\lambda^\alpha &:= \{x=(x_k)\in\omega : xy=(x_k y_k)\in \ell_1 \quad \text{for all} \quad y=(y_k)\in\lambda\},\\ \lambda^\beta &:= \{x=(x_k)\in\omega : xy=(x_k y_k)\in cs \quad \text{for all} \quad y=(y_k)\in\lambda\},\\ \lambda^\gamma &:= \{x=(x_k)\in\omega : xy=(x_k y_k)\in bs \quad \text{for all} \quad y=(y_k)\in\lambda\}.\end{aligned} \tag{1}$$

Let $A=(a_{nk})_{k,n\in\mathbb{N}}$ be an infinite matrix and $\lambda, \mu \in \omega$. We write

$$y_k=(Ax)_n=\sum_k a_{nk}x_k \tag{2}$$

and then we say that A defines a matrix transformation from λ into μ as $A:\lambda\longrightarrow\mu$ if $Ax=\{(Ax)_n\}\in\mu$ for every $x\in\lambda$. We denote the set of all infinite matrices that map the sequence space λ into the sequence space μ by (λ,μ). Thus, $A\in(\lambda,\mu)$ if and only if the right side of (2) converges for every $n\in\mathbb{N}$, that is, $A_n\in\lambda^\beta$ for all $n\in\mathbb{N}$ and we have $Ax\in\mu$ for all $x\in\lambda$. The set $\lambda_A=\{x\in\omega : Ax\in\omega\}$ is called the domain of the matrix H in X. It is known that the matrix domain λ_A is also a sequence space. The readers may refer to these nice papers [1–3] and the textbooks [4–8] concerning domain of special matrices in classical sequence spaces and the theory of summability.

If a normed sequence space λ contains a sequence (b_n) with the following property that for every $x\in\lambda$ there is a unique sequence of scalars (α_n) such that

$$\lim_{n\longrightarrow\infty}\|x-(\alpha_0 b_0+\alpha_1 b_1+\cdots+\alpha_n b_n)\|=0 \tag{3}$$

then (b_n) is called a Schauder basis for λ. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

If λ is an FK-space, $\phi \subset \lambda$ and (e^k) is a basis for λ then λ is said to have AK property, where e^k is a sequence whose only term in k^{th} place is 1 the others are zero for each $k \in \mathbb{N}$ and $\phi = span\{e^k\}$. If ϕ is dense in λ, then λ is called AD-space, thus AK implies AD.

The sequence space h defined by

$$h := \left\{ x = (x_k) \in \omega : \sum_k k|x_k - x_{k+1}| < \infty \right\} \cap c_0 \tag{4}$$

is called Hahn sequence space, named after its introducer H. Hahn [9]. The space h is a BK space with the norm

$$\|x\| = \sum_k k|x_k - x_{k+1}| + \sup_k |x_k| \text{ for all } x = (x_k) \in h. \tag{5}$$

Rao [10] proved that the space h is a BK space with AK with respect to the norm

$$\|x\|_h = \sum_k k|x_k - x_{k+1}| \text{ for all} x = (x_k) \in h. \tag{6}$$

Later on Goes [11] introduced a generalised Hahn sequence space h^d,$d = (d_k) \in \omega$ with $d_k \neq 0$ for all k, defined by

$$h^d := \left\{ x = (x_k) \in \omega : \sum_k |d_k||x_k - x_{k+1}| < \infty \right\} \cap c_0. \tag{7}$$

Quiet recently a scientific study of a more generalized Hahn sequence space h_d, is carried out by Malkowsky et al. [12], defined as follows:

$$h_d := \left\{ x = (x_k) \in \omega : \sum_k d_k|x_k - x_{k+1}| < \infty \right\} \cap c_0. \tag{8}$$

The authors proved that the space h_d is a BK space with AK with respect to the norm

$$\|x\|_{h_d} = \sum_k d_k|x_k - x_{k+1}| \text{ for all } \mathrm{x} = (\mathrm{x_k}) \in h_d, \tag{9}$$

where $d = (d_k)$ is an unbounded and monotonic increasing sequence of positive reals. Besides, the authors stated and proved various significant results concerning characterization of matrix transformations between the space h_d and classical BK spaces, and characterization of compact operators on the space h_d using Hausdorff measure of noncompactness. We refer to [2, 10, 13–23] and the survey paper [22] for more studies and results related to Hahn sequence space.

2. The New Generalized Hahn Sequence Space h_d^p

In this section, we introduce the new generalized Hahn sequence space h_d^p as follow

$$h_d^p = \left\{ x \in \omega : \sum_{k=1}^{\infty} |d_k \Delta x_k|^p < \infty \right\} \cap c_0 \tag{10}$$

where $(d_k)_{k=1}^{\infty}$ is an unbounded monotone increasing sequence of positive real numbers with $d_k \neq 0$ for all $k \in \mathbb{N}$, $1 < p < \infty$, and $\Delta x_k = x_k - x_{k+1}$.

Remark 1. It is clear that if we consider $p = 1$, then h_d^p becomes h_d.

If we consider $y_k = d_k(x_k - x_{k+1}) = (Ex)_k$ where

$$e_{nk} = \begin{cases} d_n & , k = n, \\ -d_n & , k = n+1 \\ 0 & , otherwise. \end{cases} \tag{11}$$

Moreover, if we write the terms of $y = (y_k)$ starting from k to m as follow

$$\begin{aligned} \frac{y_k}{d_k} &= x_k - x_{k+1} \\ \frac{y_{k+1}}{d_{k+1}} &= x_{k+1} - x_{k+2} \\ \frac{y_{k+2}}{d_{k+2}} &= x_{k+2} - x_{k+3} \\ &\vdots \\ \frac{y_m}{d_m} &= x_m - x_{m+1} \end{aligned} \tag{12}$$

and then when we get sum $\sum_{j=k}^{m} (y_j/d_j) = x_k - x_{m+1}$. Then we have $\sum_{j=k}^{\infty} y_j/d_j = x_k$ whenever $x_{m+1} \longrightarrow 0 (m \longrightarrow \infty)$. Equivalently, the sequence $x = (x_k)$ may also represented by $x = Gy$ where the matrix $G = (g_{nk})$ is defined by

$$g_{nk} = \begin{cases} \frac{1}{d_k} & , k \geq n, \\ 0 & , 0 < k < n. \end{cases} \tag{13}$$

Clearly $G = E^{-1}$.

Theorem 2. *The new Hahn sequence space h_d^p is a linear complete normed space space with the norm defined as*

$$\|x\|_{h_d^p} = \left(\sum_{k=1}^{\infty} |d_k \Delta x_k|^p \right)^{1/p}. \tag{14}$$

Proof. The linearity is clear. Now, we observe that the $(h_d^p, \|x\|_{h_d^p})$ is a Banach space.

(N1) if

$$\|x\|_{h_d^p} = 0 \Rightarrow \left(\sum_{k=1}^{\infty} |d_k \Delta x_k|^p\right)^{1/p} = 0 \Rightarrow d_k \Delta x_k = 0, \quad (15)$$

since $d_k \neq 0$ for each $k \in \mathbb{N}$, $\Delta x_k = 0$, that is, $x_k = x_{k+1}$ for all k, and $x \in c_0$ implies $x_k = 0$ for all k, that is, $x = 0$.

(N2) For every $x \in h_d^p$ and $\lambda \in \mathbb{K}$,

$$\|\lambda x\|_{h_d^p} = \left(\sum_{k=1}^{\infty} |d_k \Delta (\lambda x_k)|^p\right)^{1/p} = |\lambda| \left(\sum_{k=1}^{\infty} |d_k \Delta x_k|^p\right)^{1/p} = |\lambda| \|x\|_{h_d^p} \quad (16)$$

(N3) For every $x, y \in h_d^p$,

$$\|x + y\|_{h_d^p} = \left(\sum_{k=1}^{\infty} |d_k \Delta (x_k + y_k)|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |d_k \Delta x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |d_k \Delta y_k|^p\right)^{1/p} = \|x\|_{h_d^p} + \|y\|_{h_d^p}. \quad (17)$$

Now we should show that the space $(h_d^p, \|x\|_{h_d^p})$ is complete. Suppose that $x = (x^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in the space h_d^p. Therefore, for every $\epsilon > 0$, there exists a natural number $N \in \mathbb{N}$ such that

$$\left\|x^{(n)} - x^{(m)}\right\|_{h_d^p} < \frac{\epsilon}{2}, \forall m, n \geq N. \quad (18)$$

Since the Cauchy sequence $x = (x^{(n)})_{n=1}^{\infty}$ is in the space h_d^p, we obtain

$$\sum_{k=1}^{\infty} \left|d_k \Delta x_k^{(n)}\right|^p < \infty, \text{and} \quad x_k^{(n)} \longrightarrow 0 (k \longrightarrow \infty). \quad (19)$$

Now we can write the following inequality for fixed $\ell \geq k$, $m, n \geq N$ by considering the inequality

$$|a + b|^p \leq 2^p \left(|a|^p + |b|^p\right) \quad (20)$$

for $1 < p < \infty$ that

$$\begin{aligned}\left|x_k^{(n)} - x_k^{(m)}\right|^p &= \left|\sum_{j=k}^{\ell} \left(\Delta x_k^{(n)} - \Delta x_k^{(m)}\right) + \left(x_{k+1}^{(n)} - x_{k+1}^{(m)}\right)\right|^p \\ &\leq \left|\sum_{j=k}^{\ell} d_k \Delta \left(x_k^{(n)} - x_k^{(m)}\right) + \left(x_{k+1}^{(n)} - x_{k+1}^{(m)}\right)\right|^p \\ &\leq 2^p \left(\sum_{j=k}^{\ell} \left|d_k \Delta \left(x_k^{(n)} - x_k^{(m)}\right)\right|^p + \left|x_{k+1}^{(n)}\right|^p + \left|x_{k+1}^{(m)}\right|^p\right).\end{aligned} \quad (21)$$

Then we have the following result since $x^{(n)}, x^{(m)} \in c_0$ for all $m, n \in \mathbb{N}$

$$\left|x_k^{(n)} - x_k^{(m)}\right| \leq 2 \left\|x^{(n)} - x^{(m)}\right\|_{h_d^p} + \left|x_{k+1}^{(n)}\right| + \left|x_{k+1}^{(m)}\right| < \epsilon \quad (22)$$

Therefore, $x = (x^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence of Complex numbers for each $k \in \mathbb{N}$. Since $\mathbb{C}$ is complete, then $x = (x^{(n)})_{n=1}^{\infty}$ converges to an arbitrary sequence $x = (x_k) \in \mathbb{C}$ as $n \longrightarrow \infty$, that is, for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$\left|x_k^{(n)} - x_k\right| < \epsilon \quad (23)$$

for each $k \in \mathbb{N}$. Therefore, we have the following with an arbitrary ℓ that

$$\left(\sum_{k=1}^{\ell} \left|d_k \Delta \left(x_k^{(n)} - x_k^{(m)}\right)\right|^p\right)^{1/p} \leq \left\|x^{(n)} - x^{(m)}\right\|_{h_d^p} < \epsilon \quad (24)$$

hence, by letting limit as $m \longrightarrow \infty$ over (24) we obtain

$$\left(\sum_{k=1}^{\ell} \left|d_k \Delta \left(x_k^{(n)} - x_k\right)\right|^p\right)^{1/p} \leq \left\|x^{(n)} - x\right\|_{h_d^p} < \epsilon \quad (25)$$

Moreover, we obtain by (23) for any $n \geq N_0$ and $1 < p < \infty$

$$0 \leq |x_k|^p \leq 2^p \left(\left|x_k - x_k^{(n)}\right|^p + \left|x_k^{(n)}\right|^p\right) \longrightarrow 0 (k \longrightarrow \infty) \quad (26)$$

which says that $x = (x_k) \in c_0$. Furthermore,

$$\begin{aligned}\|x\|_{h_d^p} &= \left(\sum_{k=1}^{\infty} |d_k \Delta x_k|^p\right)^{1/p} = \left(\sum_{k=1}^{\infty} \left|d_k \Delta \left(x_k - x_k^{(n)} + x_k^{(n)}\right)\right|^p\right)^{1/p} \\ &\leq \left(\sum_{k=1}^{\infty} \left|d_k \Delta \left(x_k - x_k^{(n)}\right)\right|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} \left|d_k \Delta x_k^{(n)}\right|^p\right)^{1/p} < \infty.\end{aligned} \quad (27)$$

So, it shows that $x \in h_d^p$. □

Corollary 3. *The Hahn sequence space h_d^p is a BK space.*

Theorem 4. *The space h_d^p does not have AK property.*

Proof. If the sequence space h_d^p has AK property, then every sequence $x=(x_k)_{k=1}^{\infty}\in h_d^p$ is the limit of its $m-$section $x^{[m]}=\sum_{k=1}^{m}x_k e^{(k)}$, i.e.,

$$x=\lim_{m\longrightarrow\infty}x^{[m]}=\lim_{m\longrightarrow\infty}\sum_{k=1}^{m}x_k e^{(k)}=\sum_{k=1}^{\infty}x_k e^{(k)}. \tag{28}$$

Let us consider $d_k=k$ and $x_k=1/k$ for all $k\in\mathbb{N}$. Then obviously $x=(x_k)\in c_0$ and, since $p>1$,

$$\sum_{k=1}^{\infty}(d_k|x_k-x_{k+1}|)^p=\sum_{k=1}^{\infty}\left(k\left|\frac{1}{k}-\frac{1}{k+1}\right|\right)^p=\sum_{k=1}^{\infty}\left(\frac{1}{k+1}\right)^p<\infty, \tag{29}$$

that is, $x\in h_d^p$. But, if we write $x^{[m]}=\sum_{k=1}^{m}x_k e^{(k)}$ for every $m\in\mathbb{N}$ as the $m-$section of the sequence x, then we obtain for all $m\in\mathbb{N}$

$$\left\|x-x^{[m]}\right\|_{h_d^p}^p=(m|x_{m+1}|)^p+\sum_{k=m+1}^{\infty}(d_k|x_k-x_{k+1}|)^p\geq\left(\frac{m}{m+1}\right)^p. \tag{30}$$

Hence, $x^{[m]}$½x as $m\longrightarrow\infty$. It completes the proof. □

Theorem 5. *The space h_d^p is linearly isomorphic to the space ℓ_p.*

Proof. Let us define the mapping $T:h_d^p\longrightarrow\ell_p$ by $y=Tx=Ex$. Linearity of T is obvious. Since the triangle E is the matrix representation of T, therefore T is invertible too. Now, let $y\in\ell_p$ and consider the following equality:

$$\|x\|_{h_d^p}=\left(\sum_{k=1}^{\infty}|d_k\Delta x_k|^p\right)^{1/p}=\left(\sum_{k=1}^{\infty}|y_k|^p\right)^{1/p}=\|y\|_{\ell_p}<\infty. \tag{31}$$

This implies that T is onto and preserves the norm. Consequently T defines an isomorphism from h_d^p to ℓ_p. This completes the proof. □

Theorem 6. *Let us define a sequence $q^{(k)}(d)=\{q_n^{(k)}(d)\}_{n\in\mathbb{N}}$ of elements of the space h_d^p for every fixed $k\in\mathbb{N}$ by*

$$q_n^{(k)}(d)=\begin{cases}\dfrac{1}{d_k} & ,k\geq n,\\ 0 & ,1\leq k\leq n-1.\end{cases} \tag{32}$$

Then the sequence $\{q^{(k)}(d)\}_{k\in\mathbb{N}}$ is a basis for the space h_d^p and any $x\in h_d^p$ has unique representation of the form

$$x=\sum_k\lambda_k q^{(k)}(d), \tag{33}$$

where $\lambda_k=(Ex)_k$ for all $k\in\mathbb{N}$ and $1<p<\infty$.

3. $\alpha-$, $\beta-$ And $\gamma-$ Duals of h_d^p

In this section, we first calculate the $\gamma-$dual of the space h_d and then we give some basic lemmas to prove the $\alpha-$, $\beta-$ and $\gamma-$ duals of the space h_d^p. The following calculations on the $\beta-$dual of the space h_d has been stated by Goes ([11], 4.1 Theorem, p. 485) and it has recently been proved by Malkowsky et al. [12], Proposition 2.3., p.5) as

$$\{h_d\}^{\beta}=bs_d=\left\{a\in\omega:\sup_n\frac{1}{d_n}\left|\sum_{k=1}^{n}a_k\right|<\infty\right\}. \tag{34}$$

Moreover, Goes ([11], 4.2 Corolary (b)) showed the $\alpha-$dual of the space h_d as follows:

$$\{h_d\}^{\alpha}=\left\{a\in\omega:\sup_n\frac{1}{d_n}\sum_{k=1}^{n}|a_k|<\infty\right\}. \tag{35}$$

Now we can calculate the $\gamma-$dual fo the space h_d.

Lemma 7. $\{h_d\}^{\gamma}=\{a\in\omega:\sup_n(1/d_n)|\sum_{k=1}^{n}a_k|<\infty\}$.

Proof. Suppose that $a=(a_k)\in\omega$ and $x=(x_k)\in h_d$ be given. Then we have the $n^{th}-$partial sum of the series $\sum_k a_k x_k$ as

$$z_n=\sum_{k=1}^{n}a_k x_k=\sum_{k=1}^{n}a_k\left(\sum_{j=k}^{n}\frac{y_j}{d_j}\right)=\sum_{k=1}^{n-1}\left(\sum_{j=1}^{k}\frac{a_j}{d_k}\right)y_k+y_n(d_n)^{-1}\sum_{j=1}^{n}a_j=(By)_n, \tag{36}$$

where the infinite matrix $B=(b_{nk})$ is defined as

$$b_{nk}=\begin{cases}\dfrac{1}{d_k}\displaystyle\sum_{j=1}^{k}a_j & ,k\geq n,\\ 0 & ,k<n.\end{cases} \tag{37}$$

It gives that $ax\in bs$, that is, $z\in\ell_\infty$ whenever $x\in h_d$ if and only if $By\in\ell_\infty$ whenever $y\in\ell_1$. Therefore, $a\in\{h_d\}^{\gamma}$ if and only if $B\in(\ell_1,\ell_\infty)$. By Lemma 10(d) and since $\|x\|_{h_d}=\|y\|_{\ell_1}=\sum_k|y_k|<\infty$ we have that

$$\sup_k\frac{1}{d_k}\left|\sum_{j=1}^{k}a_j\right|<\infty. \tag{38}$$

This completes the proof. □

Now we have the following useful Lemmas for the further computations.

Lemma 8 (see [12], Theorem 11., p.8). *We have*

(a) $A \in (h_d, \ell_\infty)$ *if and only if*

$$\|A\|_{(h_d,\ell_\infty)} = \sup_n \|A_n\|_{bs_d} = \sup_{n,m} \frac{1}{d_m} \left| \sum_{k=1}^{m} a_{nk} \right| < \infty. \quad (39)$$

(b) $A \in (h_d, c)$ *if and only if (39) holds, and the limits*

$$\alpha_k = \lim_{n \longrightarrow \infty} a_{nk} \quad \text{exist for all } k. \quad (40)$$

Lemma 9 (see [12], Theorem 13., p.9). *We have* $A \in (h_d, \ell_1)$ *if and only if*

$$\|A\|_{(h_d,\ell_1)} = \sup_m \left(\frac{1}{d_m} \sum_{n=1}^{\infty} \left| \sum_{k=1}^{m} a_{nk} \right| \right) < \infty. \quad (41)$$

Lemma 10 (see [4], Theorem 9.7.3, p.356). *Let* $1 < p < \infty$, $q = p/(p-1)$. *Then we have*

(a) $A \in (\ell_p, \ell_1)$ *if and only if*

$$\sup_{N \in \mathbb{N} Finite} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} a_{nk} \right|^q \right) < \infty. \quad (42)$$

(b) $A \in (\ell_p, c)$ *if and only if (40) holds and*

$$\sup_n \sum_k |a_{nk}|^q < \infty. \quad (43)$$

(c) $A \in (\ell_p, \ell_\infty)$ *if and only if (43) holds*

(d) $A \in (\ell_1, \ell_\infty)$ *if and only if*

$$\sup_{n,k} |a_{nk}| < \infty. \quad (44)$$

Theorem 11. *Let* $1 < p < \infty$, $q = p/(p-1)$. *Then*

$$\{h_d^p\}^\alpha = \left\{ a = (a_k) \in \omega : \sup_{N \in \mathbb{N} finite} \sum_k \left| \sum_{n \in N} \frac{a_n}{d_k} \right|^q < \infty \right\}. \quad (45)$$

Proof. Suppose that $a = (a_n) \in \omega$ be given and consider the following equation that

$$a_n x_n = a_k \sum_{k=n}^{\infty} \frac{y_k}{d_k} = \sum_{k=n}^{\infty} \frac{a_n}{d_k} y_k = (Dy)_n, n \in \mathbb{N}, \quad (46)$$

where the matrix $D = (d_{nk})$ is defined as

$$d_{nk} = \begin{cases} \dfrac{a_n}{d_k} & , k \geq n, \\ 0 & , k < n \end{cases} \quad (47)$$

for all $k, n \in \mathbb{N}$. It gives us that $ax = (a_n x_n) \in \ell_1$ whenever $x \in h_d^p$ if and only if $Dy \in \ell_1$ whenever $y = (y_n) \in \ell_p$. Therefore, $a = (a_n) \in \{h_d^p\}^\alpha$ whenever $x \in h_d^p$ if and only if $D \in (\ell_p, \ell_1)$. Therefore, the condition in (42) of Lemma 10(a) holds with d_{nk} instead of a_{nk}, that is,

$$\sup_{N \in \mathbb{N} Finite} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \frac{a_n}{d_k} \right|^q \right) < \infty. \quad (48)$$

It gives the α – dual of the space h_d^p. □

Theorem 12. $\{h_d^p\}^\beta = \{a = (a_k) \in \omega : \sup_n (1/|d_n|^q) \sum_k |\sum_{j=k}^{n} a_j|^q < \infty\}$.

Proof. Suppose that $a = (a_k) \in \omega$ and $x = (x_k) \in h_d^p$ are given. Let us consider the following equality.

$$s_n = \sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left(\sum_{j=k}^{n} \frac{y_j}{d_j} \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{k} \frac{a_j}{d_k} \right) y_k = (By)_n, \quad (49)$$

where the matrix $B = (b_{nk})$ is defined as in (37) for all $k, n \in \mathbb{N}$. Then we can say that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in h_d^p$ if and only if $By \in c$ whenever $y = (y_k) \in \ell_p$. Therefore, $a = (a_k) \in \{h_d^p\}^\beta$ if and only if $B \in (\ell_p, c)$. Then the condition in Lemma 10(b) holds with b_{nk} instead of a_{nk}, that is

$$\lim_{n \longrightarrow \infty} b_{nk} \text{ exist for all } k; \qquad \sup_n \sum_k |b_{nk}|^q < \infty. \quad (50)$$

Thus, we can obtain that $\{h_d^p\}^\beta = \{a = (a_k) \in \omega : \sup_n (1/|d_n|^q) \sum_k |\sum_{j=k}^{n} a_j|^q < \infty\} \cap cs$. It is clearly seen that

$$\left\{ a = (a_k) \in \omega : \sup_n \frac{1}{|d_n|^q} \sum_k \left| \sum_{j=k}^{n} a_j \right|^q < \infty \right\} \subset cs \quad (51)$$

and then

$$\{h_d^p\}^\beta = \left\{ a = (a_k) \in \omega : \sup_n \frac{1}{|d_n|^q} \sum_k \left| \sum_{j=k}^{n} a_j \right|^q < \infty \right\} \quad (52)$$

as we desired. □

Theorem 13. $\{h_d^p\}^\gamma = \{a = (a_k) \in \omega : \sup_n (1/|d_n|^q)\sum_k |\sum_{j=k}^n a_j|^q < \infty\}$.

Proof. Since we have similar pattern with Theorem 12, we can conclude the proof with the following computation only.

Let us say that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in h_d^p$ if and only if $By \in \ell_\infty$ whenever $y = (y_k) \in \ell_p$. Therefore, $a = (a_k) \in \{h_d^p\}^\beta$ if and only if $B \in (\ell_p ; \ell_\infty)$. Then the condition in Lemma 10(c) holds with b_{nk} instead of a_{nk}, that is

$$\sup_n \sum_k |b_{nk}|^q < \infty. \tag{53}$$

Thus, we can obtain that $\{h_d^p\}^\gamma = \{a = (a_k) \in \omega : \sup_n (1/|d_n|^q)\sum_k |\sum_{j=k}^n a_j|^q < \infty\}$ as we desired. □

4. Some Matrix Transformations from and into the Space h_d

In this section, we characterize the classes (h_d, λ), where $\lambda = \{bv, bv_p, bv_\infty, bs, cs,\}$, and (μ, h_d), where $\mu = \{bv, bv_0, bs, cs_0, cs\}$.

Remark 14. If $d_k = k$ for every $k \in \mathbb{N}$, then characterization of Theorem 17-Corollary 23 yields the characterisation of (h, λ), where $\lambda = \{bv, bv_p, bv_\infty, bs, cs,\}$

Definition 15 (see [8], Definition 7.4.2). Let X be a BK space. A subset E of the set ϕ called a determining set for X if $D(X) = \bar{B}_X \cap \phi$ is the absolutely convex hull of E.

Proposition 16 (see [12], Proposition 3.2, p.8). *Let*

$$s(d,k) = \frac{1}{d_k} \cdot e^{[k]} \text{ for each } k \in \mathbb{N}, \text{ and } E = \{s(d,k) : k \in \mathbb{N}\}. \tag{54}$$

Then E is a determining set for h_d.

Theorem 17. *The infinite matrix $A = (a_{nk}) \in (h_d, bv)$ if and only if*

$$\sup_{m\in\mathbb{N}} \frac{1}{|d_m|} \sum_n \left| \sum_{k=1}^m (a_{nk} - a_{n-1,k}) \right| < \infty. \tag{55}$$

Proof. Suppose that $E = \{(1/d_m)e^{[m]} : m \in \mathbb{N}\}$ is a determining set for the sequence space h_d. The sequence space bv is BK space with $\|.\|_{bv}$. Let $y^{(m)} = (1/d_m)e^{[m]} \in E$. Then

$$A_n y^{(m)} = \sum_{k=1}^\infty a_{nk} y_k^{(m)} = \frac{1}{d_m} \sum_{k=1}^m a_{nk} \text{ for all } n = 1, 2, \cdots \tag{56}$$

Hence,

$$\begin{aligned} \left\| A_n y^{(m)} \right\|_{bv} &= \sum_n \left| A_n y^{(m)} - A_{n-1} y^{(m)} \right| = \sum_n \left| \frac{1}{d_m} \sum_{k=1}^m a_{nk} - \frac{1}{d_m} \sum_{k=1}^m a_{n-1,k} \right| \\ &= \frac{1}{|d_m|} \sum_n \left| \sum_{k=1}^m (a_{nk} - a_{n-1,k}) \right| < \infty \end{aligned} \tag{57}$$

which gives the condition in (55). □

Proposition 18 (see [1], Lemma 5.3). *Let X, Y be any two sequence spaces, A be an infinite matrix and U a triangle matrix. Then $A \in (X, Y_U)$ if and only if $UA \in (X, Y)$.*

Corollary 19. *The infinite matrix $A = (a_{nk}) \in (h_d, bv_p) = (h_d, (\ell_p)_\Delta)$ if and only if*

$$\sup_{m\in\mathbb{N}} \frac{1}{|d_m|^p} \sum_n \left| \sum_{k=1}^m (a_{nk} - a_{n-1,k}) \right|^p < \infty, (1 \le p < \infty). \tag{58}$$

Corollary 20. *The infinite matrix $A = (a_{nk}) \in (h_d, bv_\infty) = (h_d, (\ell_\infty)_\Delta)$ if and only if*

$$\sup_{m,n\in\mathbb{N}} \frac{1}{|d_m|} \sum_n \left| \sum_{k=1}^m (a_{nk} - a_{n-1,k}) \right| < \infty. \tag{59}$$

Theorem 21. *The infinite matrix $A = (a_{nk}) \in (h_d, bs)$ if and only if*

$$\sup_{m,n\in\mathbb{N}} \frac{1}{|d_m|} \left| \sum_{k=1}^n \sum_{j=1}^m a_{kj} \right| < \infty. \tag{60}$$

Proof. The proof is similar to that of Theorem 17 with the norm $\|.\|_{bs}$ replaced by the norm $\|.\|_{bv}$. Therefore, we have only the following norm

$$\begin{aligned} \left\| A_n y^{(m)} \right\|_{bs} &= \sup_{m\in\mathbb{N}} \left| \sum_{k=1}^n A_k y^{(m)} \right| = \sup_{m\in\mathbb{N}} \left| \sum_{k=1}^n \frac{1}{|d_m|} \sum_{j=1}^m a_{kj} \right| \\ &= \sup_{m\in\mathbb{N}} \frac{1}{|d_m|} \left| \sum_{k=1}^n \sum_{j=1}^m a_{kj} \right|. \end{aligned} \tag{61}$$

□

If we take supremum $n \in \mathbb{N}$ from both sides of above equality, then we can easily see the condition in (60).

Theorem 22. *The infinite matrix $A = (a_{nk}) \in (h_d, cs)$ if and only if the condition in (60) holds and*

$$\lim_{n\to\infty} \sum_{k=1}^n a_{nk} = \alpha_k \text{ exists for each } k \in \mathbb{N}. \tag{62}$$

Proof. Suppose that $A = (a_{nk}) \in (h_d, cs)$. Then Ax exists and is in cs for each $x = (x_k) \in h_d$. Thus, the partial sum of the series Ax converges, that is the condition in (62) is immediate. The rest of the proof is similar to that of Theorem 17 by replacing the norm $\|.\|_{bv}$ with the norm $\|.\|_{bs} = \|.\|_{cs}$. Hence, we obtain the necessity of the condition in (4.4). □

Corollary 23. *The infinite matrix $A = (a_{nk}) \in (h_d, cs_0)$ if and only if the conditions in (60) and (62) hold with $\alpha_k = 0$ for every $k \in \mathbb{N}$.*

Theorem 24. *The infinite matrix $A = (a_{nk}) \in (bv_0, h_d)$ if and only if*

$$\lim_{n \longrightarrow \infty} \sum_{k=1}^{m} a_{nk} = 0 \quad (63)$$

$$\sup_{k \in \mathbb{N}} \sum_{n=1}^{\infty} |d_n| \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right| < \infty. \quad (64)$$

Proof. Suppose that $E = \{e^{[m]} : m \in \mathbb{N}\}$ be a determining set of bv_0 and $y^{(m)} = e^{[m]} \in E$ for each $m \in \mathbb{N}$. Then the k^{th} column of $Ay^{(m)}$ where

$$A^k y^{(m)} = \sum_{k=1}^{\infty} a_{nk} y_k^{(m)} = \sum_{k=1}^{m} a_{nk}, \text{for} \quad n = 1, 2, \cdots \quad (65)$$

Then,

$$\left\| A^k y^{(m)} \right\|_{h_d} = \sum_{n=1}^{\infty} |d_n| \left| \sum_{k=1}^{m} (a_{nk} - a_{n+1,k}) \right|, \quad \text{for } all\, k \in \mathbb{N}. \quad (66)$$

□

Hence, the sequence $\{A^k y^{(m)}\}_{k=1}^{\infty}$ is bounded in h_d if and only if $\sup_k \|A^k y^{(m)}\|_{h_d} < \infty$ which shows the necessity of the condition in (64). Moreover, $A^k y^{(m)} \in c_0$ for all $k \in \mathbb{N}$, then the condition in (63) is also immediate.

Corollary 25. *The infinite matrix $A = (a_{nk}) \in (bv, h_d)$ if and only if the conditions in (63) and (64) hold, and*

$$Ae \in h_d \quad (67)$$

Theorem 26. *The infinite matrix $A = (a_{nk}) \in (bs, h_d)$ if and only if*

$$\sup_{K \in \mathbb{N} finite} \sum_{k=1}^{\infty} \left| \sum_{n \in K} d_n (b_{nk} - b_{n+1,k}) \right|, \quad (68)$$

$$\lim_{n \longrightarrow \infty} \sum_{k=1}^{\infty} |b_{nk}| = 0 \quad (69)$$

where the matrix $B = (b_{nk})$ is defined as $b_{nk} = a_{nk} - a_{n,k+1}$.

Proof. Suppose that $A = (a_{nk}) \in (bs, h_d)$ and Ax exists and is in h_d for every $x \in bs$. Thus, $A_n \in bs^{\beta} = bv_0 \subset c_0$. Let x be a sequence such that $v_k = \sum_{i=1}^{k} x_i$. If $v \in \ell_{\infty}$ and $x_n = v_n - v_{n-1}$, then $x \in bs$. By ([8], Lemma 8.5.3(i)), we can say that $Ax = Bv$. It says that $B \in (\ell_{\infty}, h_d)$. So the necessity of the conditions in (68) and (69) are immediate. □

Theorem 27. *The infinite matrix $A = (a_{nk}) \in (cs_0, h_d)$ if and only if the condition in (68) holds and*

$$\lim_{n \longrightarrow \infty} b_{nk} = 0 \quad (70)$$

where the matrix $B = (b_{nk})$ is defined as $b_{nk} = a_{nk} - a_{n,k+1}$.

Proof. The proof is similar to that of Theorem 26 with $A_n \in cs_0^{\beta} = cs^{\beta} = bv \subset c$ and by ([8], Lemma 8.5.3(i)), we can say that $B \in (c_0, h_d)$. So the necessity of the conditions in (68) and (70) are immediate. □

Corollary 28. *The infinite matrix $A = (a_{nk}) \in (cs, h_d)$ if and only if the conditions in (67), (68), (70) hold.*

Remark 29. If we consider $d_k = k$ for every $k \in \mathbb{N}$, then the characterization of Theorem 17, Corollary 19-Corollary 23 yields the characterisation of (h, λ), where $\lambda = \{bv, bv_p, bv_{\infty}, bs, cs,\}$, and the characterization of Theorem 24-Corollary 28 yields the characterisation of(μ, h), where $\mu = \{bv, bv_0, bs, cs_0, cs\}$ (see [18], p.13-16).

5. Matrix Transformations on the Hahn Sequence Space h_d^p

In this section, we characterize matrix $(h_d^p ; \lambda)$ and $(\mu ; h_d^p)$ where $\lambda = \{\ell_{\infty}, c, c_0, \ell_1, h_d, bv, bs, cs\}$ and $\mu = \{\ell_1, bv_0, bs, cs_0\}$, respectively. The following lemma is significant for our investigation:

Lemma 30 (see [1]). *Matrix transformation between BK-spaces are continuous.*

Theorem 31. *$A = (a_{nk}) \in (h_d^p, \ell_{\infty})$ if and only if*

$$\sup_{n,m} \frac{1}{|d_m|^q} \sum_{k=1}^{\infty} \left| \sum_{j=k}^{m} a_{nj} \right|^q < \infty, \quad (71)$$

$$\sup_{n} \sum_{k=1}^{\infty} \frac{1}{|d_k|^q} \left| \sum_{j=1}^{k} a_{nj} \right|^q < \infty. \quad (72)$$

Proof. Let $1 < p < \infty$ and $A \in (h_d^p, \ell_{\infty})$. This means that Ax exists and is contained in the space ℓ_{∞} for each $x \in h_d^p$. Apparently $(a_{nk})_{k=1}^{\infty} \in \{h_d^p\}^{\beta}$. This establishes the necessity of condition in (71).

We recall that $y_k = d_k \Delta x_k$ or equivalently $x_k = \sum_{j=k}^{\infty} y_j/d_j$ for each $k \in \mathbb{N}$. This leads us to the following equality

$$\sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} \sum_{j=k}^{\infty} \frac{y_j}{d_j} = \sum_{k=1} a_{nk} \sum_{j=k}^{\infty} \Delta x_j = \sum_{j=1}^{\infty} \sum_{k=1}^{j} a_{nk} \frac{y_j}{d_j} \quad (73)$$

for each $n \in \mathbb{N}$. Since h_d^p and ℓ_∞ are *BK* spaces, Lemma 30 ascertains that there exists a positive number $M < \infty$ such that

$$\|Ax\|_{\ell_\infty} \le M \|x\|_{h_d^p} \quad (74)$$

for all $x \in h_d^p$. Therefore, by the aid of Hölder's inequality together with (73) and Theorem 5, we get

$$\frac{\|Ax\|_{\ell_\infty}}{\|y\|_{\ell_p}} = \sup_n \frac{\left|\sum_{k=1}^{\infty} \sum_{j=1}^{k} a_{nj} (y_k/d_k)\right|}{\|y\|_{\ell_p}} \le \sup_n \left(\sum_{k=1}^{\infty} \frac{1}{|d_k|^q} \left| \sum_{j=1}^{k} a_{nj} \right|^q \right)^{1/q} < \infty \quad (75)$$

for all $x \in h_d^p$. This establishes the necessity of condition in (72).

Conversely, assume that conditions in (71) and (72) hold and take any $x \in h_d^p$. Then $(a_{nk})_{k=1}^{\infty} \in \{h_d^p\}^\beta$ which implies that Ax exists. Again applying Hölder's inequality and the fact that $y = (y_k) \in \ell_p$, we deduce that

$$\|Ax\|_{\ell_\infty} = \sup_n \left| \sum_{k=1}^{\infty} \sum_{j=1}^{k} a_{nj} \frac{y_k}{d_k} \right| \le \sup_n \left(\sum_{k=1}^{\infty} \frac{1}{|d_k|^q} \left| \sum_{j=1}^{k} a_{nj} \right|^q \right)^{1/q} \left(\sum_{k=1} |y_k|^p \right)^{1/p} < \infty. \quad (76)$$

Consequently, $A \in (h_d^p, \ell_\infty)$. This completes the proof. □

Theorem 32. *$A \in (h_d^p, c)$ if and only if (71) and (72) hold, and for each $k \in \mathbb{N}$, there exists $\alpha_k \in \mathbb{R}$ such that*

$$\lim_{n \longrightarrow \infty} \frac{1}{d_k} \sum_{j=1}^{k} a_{nj} = \alpha_k. \quad (77)$$

Proof. Let $A \in (h_d^p, c)$. Then Ax exists and is contained in the space c for all $x \in h_d^p$. Since $c \subset \ell_\infty$, $Ax \in \ell_\infty$. Thus the necessity of conditions in (71) and (72) follows from Theorem 31. Let us consider $x = Ge^{(k)}$ and define the matrix $D = (d_{nk})$ by

$$d_{nk} = \frac{1}{d_k} \sum_{j=1}^{k} a_{nj} \quad (78)$$

for all $n, k = 1, 2, 3, \cdots$. Then by virtue of equality (73), we get that

$$Ax = A\left(Ge^{(k)}\right) = D\left(E\left(Ge^{(k)}\right)\right) = De^{(k)} = (d_{nk})_{n \in \mathbb{N}}. \quad (79)$$

Since $Ax \in c$, so $(d_{nk})_{n \in \mathbb{N}} \in c$. This proves the necessity of the condition in (77).

Conversely, assume that the conditions in (71), (72) and (77) hold. Then $(a_{nk})_{k \in \mathbb{N}} \in \{h_d^p\}^\beta$ for each $n \in \mathbb{N}$ which implies that Ax exists for all $x \in h_d^p$. Therefore we again obtained equality (73). Then, one can notice that the conditions in (71) and (77) correspond to the conditions in (43) and (40), respectively, with d_{nk} instead of a_{nk}. This concludes that $Dy = Ax \in c$. Hence $A \in (h_d^p, c)$. □

Replacing c by c_0 in Theorem 32, we obtain the following corollary:

Corollary 33. *$A \in (h_d^p, c_0)$ if and only if (71) and (72) hold, and (77) also holds with $\alpha_k = 0$ for all k.*

Theorem 34. *$A = (a_{nk}) \in (h_d^p, \ell_1)$ if and only if (71) holds, and*

$$\sup_{N \in \mathbb{N} Finite} \sum_{k=1}^{\infty} \left| \sum_{n \in N} \frac{1}{d_k} \sum_{j=1}^{k} a_{nj} \right|^q < \infty, 1 < p < \infty. \quad (80)$$

Proof. This is similar to the proof of Theorem 31. Hence details are omitted. □

Theorem 35. *$A = (a_{nk}) \in (h_d^p, h_d)$ if and only if the condition in (71) holds and*

$$\lim_{n \longrightarrow \infty} \frac{1}{d_k} \sum_{j=1}^{k} a_{nj} = 0, \quad \text{for all} \quad k \in \mathbb{N} \quad (81)$$

$$\sup_{k \in \mathbb{N}} \frac{1}{|d_k|} \left(\sum_{n=1}^{\infty} \left| d_n \sum_{j=1}^{k} \left(a_{nj} - a_{n+1,j} \right) \right|^q \right)^{1/q} < \infty, (1 < p < \infty). \quad (82)$$

Proof. Let $1 < p < \infty$ and suppose that $A \in (h_d^p, h_d)$ such that Ax exists and is in the space h_d for each $x \in h_d^p$. Thus, we clerly see that $(a_{nk})_{k=1}^{\infty} \in \{h_d^p\}^\beta$ which proves the necessity of condition in (71).

By using the relation $y_k = d_k \Delta x_k$ or equivalently $x_k = \sum_{j=k}^{\infty} y_j/d_j$ for each $k \in \mathbb{N}$ between the terms of the sequences $x = (x_k)$ and $y = (y_k)$, we reach the equality in (73) which can be written as in the following

$$\sum_{k=1}^{\infty} a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} \sum_{j=k}^{\infty} \frac{y_j}{d_j} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} a_{nj} \frac{y_k}{d_k} = \sum_{k=1}^{\infty} d_{nk} y_k \quad (83)$$

where the infinite matrix $D = (d_{nk})$ is defined as $d_{nk} = 1/d_k \sum_{j=1}^{k} a_{nj}$ for each $n, k \in \mathbb{N}$. Since h_d^p and h_d are *BK* spaces, we revisit Lemma 30 that there exists a positive real number

N such that

$$\|Ax\|_{h_d} \le N\|x\|_{h_d^p} \tag{84}$$

for all $x \in h_d^p$. Therefore, by using Hölder's inequality together with (73) and Theorem 5, we get

$$\begin{aligned}\frac{\|Ax\|_{h_d}}{\|x\|_{h_d^p}} &= \sup_k \frac{\left(\sum_{n=1}^{\infty}|d_n \Delta A_n x|^q\right)^{1/q}}{\|y\|_{\ell_p}} \\ &\le \sup_k \frac{1}{|d_k|}\left(\sum_{n=1}^{\infty}\left|d_n \sum_{j=1}^{k}(a_{nj}-a_{n+1,j})\right|^q\right)^{1/q} < \infty\end{aligned} \tag{85}$$

for all $x \in h_d^p$. This shows the necessity of condition in (82). Moreover, since $Ax = Dy \in h_d$ for every $x \in h_d^p$ the necessity of condition in (81) is also seen.

Conversely, suppose that the conditions in (71), (81), and (82) hold. Let us take any $x \in h_d^p$. Then $A_n \in \{h_d^p\}^{\beta}$ which implies that Ax exists. The rest of proof follows the similar path to that of Theorem 31 with the following inequality

$$\begin{aligned}\|Ax\|_{h_d} &= \sup_k \sum_{n=1}^{\infty}|d_n\Delta(A_n x)| \\ &\le \sup_k \frac{1}{|d_k|}\left(\sum_{n=1}^{\infty}\left|d_n\sum_{j=1}^{k}(a_{nj}-a_{n+1,j})\right|^q\right)^{1/q}\left(\sum_{k=1}|y_k|^p\right)^{1/p} < \infty.\end{aligned} \tag{86}$$

Thus, $A \in (h_d^p, h_d)$. This completes the proof. □

Since the proof of the following results can be done similary to that of Theorem 35, we give them without their proofs.

Corollary 36. *The followings hold.*

(i) *The infinite matrix $A = (a_{nk}) \in (h_d^p, bv)$ if and only if the condition in (71) holds and*

$$\sup_{m\in\mathbb{N}} \frac{1}{|d_m|^q}\sum_n\left|\sum_{k=1}^{m}(a_{nk}-a_{n-1,k})\right|^q < \infty \tag{87}$$

(ii) *The infinite matrix $A = (a_{nk}) \in (h_d^p, bs)$ if and only if the condition in (71) holds and*

$$\sup_{m,n\in\mathbb{N}} \frac{1}{|d_m|^q}\left|\sum_{k=1}^{n}\sum_{j=1}^{m}a_{kj}\right|^q < \infty. \tag{88}$$

(iii) *The infinite matrix $A = (a_{nk}) \in (h_d^p, cs)$ if and only if the conditions in (71) and (88) hold, and*

$$\lim_{n\to\infty}\sum_{k=1}^{m}\frac{1}{d_k}\sum_{j=1}^{k}a_{nj} = \alpha_k \quad \textit{exists for each} \quad k \in \mathbb{N}. \tag{89}$$

Theorem 37. *$A = (a_{nk}) \in (\ell_1, h_d^p)$ if and only if*

$$\lim_{n\to\infty} a_{nk} = 0, \quad \textit{for all} \quad k \in \mathbb{N}, \tag{90}$$

$$\sup_n\left(\sum_{n=1}^{\infty}\left|d_n(a_{nk}-a_{n+1,k})\right|^p\right)^{1/p} < \infty, (1 < p < \infty). \tag{91}$$

Proof. Since h_d^p is a BK space by Corollary 3, we apply ([8], Example 8.4.1) to obtain that $A \in (\ell_1, h_d^p)$ if and only if the columns of A form a bounded set in h_d^p.

We have

$$\left\|A^k\right\|_{h_d^p} = \left(\sum_{n=1}^{\infty}\left|d_n(a_{nk}-a_{n+1,k})\right|^p\right)^{1/p} \quad \text{for all} \quad k. \tag{92}$$

Hence the set $\{A^k : k \in \mathbb{N}\}$ is bounded in h_d^p if and only if $\sup_k\|A^k\|_{h_d^p} < \infty$, which gives the condition in (91). Moreover $A^k \in c_0$ for all k, which gives the condition in (87). This completes the proof. □

Now we give the following corollaries without their proofs since the proofs are followed similarly to that of Theorem 37.

Corollary 38. *The followings hold.*

(i) *The infinite matrix $A = (a_{nk}) \in (bv_0, h_d^p)$ if and only if the condition in (63) holds, and*

$$\sup_{k\in\mathbb{N}}\sum_{n=1}^{\infty}\left|d_n\sum_{k=1}^{m}(a_{nk}-a_{n+1,k})\right|^p < \infty. \tag{93}$$

(ii) *The infinite matrix $A = (a_{nk}) \in (bs, h_d^p)$ if and only if*

$$\sup_{K\in\mathbb{N}\, finite}\sum_{k=1}^{\infty}\left|\sum_{n\in K}d_n(b_{nk}-b_{n+1,k})\right|^p, \tag{94}$$

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}|b_{nk}| = 0 \tag{95}$$

where the matrix $B = (b_{nk})$ is defined as $b_{nk} = a_{nk} - a_{n,k+1}$.

(iii) *The infinite matrix $A = (a_{nk}) \in (cs_0, h_d^p)$ if and only if the condition in (94) holds and*

$$\lim_{n\longrightarrow\infty} b_{nk} = 0 \quad (96)$$

where the matrix $B = (b_{nk})$ *is defined as* $b_{nk} = a_{nk} - a_{n,k+1}$.

6. Conclusion

Most recently, Malkowsky at al [12] defined the generalized Hahn sequence space h_d, where d is an unbounded monotone increasing sequence of positive real numbers, and characterized several classes of bounded linear operators or matrix transformations from h_d into $\lambda = \{\ell_\infty, c, c_0, \ell_1, h_d\}$, and also from $\lambda = \{\ell_\infty, c, c_0, \ell_1\}$ into h_d. Moreover, the norms of the corresponding bounded linear operators and the Hausdorff measure of non-compactness for the operators in the above classes, and one application given by a tridiagonal matrix to present a Fredholm operator from h_d into itself were studied in [12].

Malkowsky [14] established the characterisations of the classes of bounded linear operators from the generalised Hahn sequence space h_d into the spaces $[c_0]$, $[c]$ and $[c_\infty]$. Moreover, he proved estimates for the Hausdorff measure of noncompactness of bounded linear operators from h_d into $[c]$, and identities for the Hausdorff measure of noncompactness of bounded linear operators from h_d to $[c_0]$, and then he used these results to characterise the classes of compact operators from h_d to $[c]$ and $[c_0]$.

Dolic′anin-Dekic′ and Gilic′ [20] established the characterisations of the classes of bounded linear operators from the generalised Hahn sequence space h_d into the spaces w_0, w and w_∞. Then they proved estimates for the Hausdorff measure of noncompactness of bounded linear operators from h_d into w, and identities for the Hausdorff measure of noncompactness of bounded linear operators from h_d into w_0, and then they used these results to characterise the classes of compact operators from h_d to w_0 and w.

In this article, we defined new generalized Hahn sequence space h_d^p, where $d = (d_k)_{k=1}^\infty$ is an unbounded monotone increasing sequence of positive real numbers with $d_k \neq 0$ for all $k \in \mathbb{N}$, and $1 < p < \infty$. Then, we proved some topological properties and showed some inclusion relations. Moreover, we calculate the $\alpha -$, $\beta -$, and $\gamma -$duals of h_d^p. Furthermore, we characterize the new matrix classes (h_d, λ), where $\lambda = \{bv, bv_p, bv_\infty, bs, cs,\}$, and (μ, h_d), where $\mu = \{bv, bv_0, bs, cs_0, cs\}$. In the last section, we prove the necessary and sufficient conditions of the matrix transformations from h_d^p into $\lambda = \{\ell_\infty, c, c_0, \ell_1, h_d, bv, bs, cs\}$, and from $\mu = \{\ell_1, bv_0, bs, cs_0\}$ into h_d^p.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author(s) declare(s) that they have no conflicts of interest.

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