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Research Article

A Weak Convergence Theorem for Common Fixed Points of Two Nonlinear Mappings in Hilbert Spaces

Takanori Ibaraki,[1] **Shunsuke Kajiba**,[2,3] **and Yukio Takeuchi**[4]

[1] *Department of Mathematics Education, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan*
[2] *Graduate School of Environment and Information Sciences, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan*
[3] *Chiba Reimei High School, 625 Ho Yachimata-shi, Chiba 289-1115, Japan*
[4] *Takahashi Institute for Nonlinear Analysis, 1-11-11 Nakazato, Minami, Yokohama 232-0063, Japan*

Correspondence should be addressed to Takanori Ibaraki; ibaraki@ynu.ac.jp

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In this paper, by using properties of attractive points, we study an iteration scheme combining simplified Baillon type and Mann type to find a common fixed point of commutative two nonlinear mappings in Hilbert spaces. Then, we apply the obtained results to prove a new weak convergence theorem.

1. Introduction

In 1963, DeMarr [1] proved a common fixed point theorem for a family of commuting nonexpansive mappings in a Banach space. After DeMarr, many researchers studied this subject (see [2–6] and others).

On the other hand, in 1975, Baillon [7] proved a mean convergence theorem known as the first nonlinear ergodic theorem in a Hilbert space. After Baillon, many researchers have studied topics related to his mean convergence theorem. In 1997, Shimizu and Takahashi [8] introduced the iteration scheme that combines Baillon type and Halpern type [9]. Then, they proved a strong convergence theorem to a common fixed point of a finite family of commutative nonexpansive mappings in Hilbert spaces. In 1998, Atsushiba and Takahashi [10] introduced the iteration scheme that combines Baillon type and Mann type [11] and proved a weak convergence theorem to a common fixed point of commutative two nonexpansive mappings in uniformly convex Banach spaces. In 2002, Suzuki [12] studied for common fixed points of commutative two nonexpansive mappings in general Banach spaces. Then, he proved a strong convergence theorem using Atsushiba and Takahashi's iteration scheme. Stimulated by Suzuki [12], Takeuchi [13] introduced a new iteration scheme combining simplified Baillon type and Mann type and proved the following strong convergence theorem in general Banach spaces.

Theorem 1 (see [13]). *Let E be a Banach space and let C be a compact convex subset of E. Let S and T be nonexpansive self-mappings on C with $ST = TS$. Let $\{a_n\}$ be a sequence in $[0, 1]$ satisfying $0 < \liminf_{n\longrightarrow\infty} a_n \le \limsup_{n\longrightarrow\infty} a_n < 1$. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{a_n}{2n}\sum_{i=1}^{n}\sum_{j=i}^{i+1} S^i T^j x_n + (1 - a_n)x_n, \tag{1}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to some common fixed point z of S and T.

Also, some researchers studied topics related to common fixed points of various nonlinear mappings or semigroups of nonlinear mappings, and some convergence theorems were proved; for example, see [14–16] and others. In addition, Baillon's theorem [7] evolved as follows. In 2011, Takahashi and Takeuchi [17] proposed the notion of an attractive point of a mapping T. They denote by $A(T)$ the set of all attractive points of T and by $P_{A(T)}$ the metric projection from H onto the closed convex set $A(T)$. An attractive point is an important notion related to fixed points (see [18–21] and therein). Then, Takahashi and Takeuchi [17] proved the following Baillon type mean convergence theorem finding an attractive point for a wide class of nonlinear mappings called generalized hybrid [22].

Theorem 2 (see [17]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a generalized hybrid mapping from C into itself. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$\begin{aligned} v_1 &\in C, \\ v_{n+1} &= T v_n, \\ b_n &= \frac{1}{n} \sum_{k=1}^{n} v_k, \end{aligned} \tag{2}$$

for each $n \in \mathbb{N}$. Suppose $\{v_n\}$ is bounded. Then, the following hold:

(1) $A(T)$ is nonempty, closed, and convex

(2) $\{b_n\}$ converges weakly to $u \in A(T)$ such that $u = \lim_{n\longrightarrow\infty} P_{A(T)} v_n$

Motivated by the works as above, considering properties of attractive points, we study the iteration scheme proposed by Takeuchi [13]. Then, using the obtained results, we prove a new weak convergence theorem for common fixed points of commutative two nonlinear mappings in Hilbert spaces.

2. Preliminaries

We present some of fundamental concepts and some symbols used throughout this paper. We denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{N}$ the set of all positive integers, and by $\mathbb{N}_0$ the set of all nonnegative integers. Also, we denote by $\mathbb{N}_i$ the set $\{k \in \mathbb{N}_0 : i \le k\}$ for each $i \in \mathbb{N}_0$ and by $\mathbb{N}(i, j)$ the set $\{k \in \mathbb{N}_0 : i \le k \le j\}$ for each $i, j \in \mathbb{N}_0$ with $i \le j$. Obviously, $\mathbb{N}_1 = \mathbb{N}$.

H always denotes a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H. Sometimes the strong convergence and weak convergence of $\{x_n\}$ to a point $x \in H$ are denoted by $x_n \longrightarrow x$ and $x_n \rightharpoonup x$, respectively. So, $x_n \longrightarrow x$ implies $\lim_{n\longrightarrow\infty} \|x_n - x\| = 0$, and $x_n \rightharpoonup x$ implies $\lim_{n\longrightarrow\infty} \langle x_n - x, y \rangle = 0$ for each $y \in H$. Then, we know the following basic facts:

(i) A nonempty closed convex subset C of H is weakly closed

(ii) A bounded sequence of H has a weakly convergent subsequence

(iii) $x_n \rightharpoonup z \in H$ if every weak cluster point of $\{x_n\}$ and z are the same

A Hilbert space H has the Opial property; that is, if a sequence $\{u_n\}$ in H converges weakly to a point $u \in H$, then

$$\liminf_{n\longrightarrow\infty} \|u_n - u\| < \liminf_{n\longrightarrow\infty} \|u_n - v\|, \tag{3}$$

for all $v \in H$ with $u \neq v$.

Let C be a nonempty subset of H and let T be a mapping from C into H. Sometimes, we denote by T^0 the identity mapping I on C. Then, we denote by $F(T)$ the set of all fixed points of T and by $A(T)$ the set of all attractive points of T, that is,

$$\begin{aligned} F(T) &= \{x \in C : x = Tx\}, \\ A(T) &= \{y \in H : \|Tx - y\| \le \|x - y\| \text{ for all } x \in C\}. \end{aligned} \tag{4}$$

T is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We say that $I - T$ is demiclosed at 0 if $z \in F(T)$ holds whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ for some $z \in C$ and $x_n - Tx_n \longrightarrow 0$. Then, $I - T$ is demiclosed at 0 if T is nonexpansive. T is called quasi-nonexpansive if $\emptyset \neq F(T) \subset A(T)$. It is easy to see that a nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive. Aoyama et al. [23] proposed λ-hybrid mappings for $\lambda \in \mathbb{R}$. T is called λ–hybrid if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2(1 - \lambda)\langle x - Tx, y - Ty \rangle, \tag{5}$$

for all $x, y \in C$. We can easily verify that a mapping of several important classes of nonlinear mappings is λ-hybrid for some $\lambda \in \mathbb{R}$. For example, a nonspreading mapping [24] is 0-hybrid; a hybrid mapping [25] is 1/2-hybrid and a nonexpansive mapping is 1-hybrid. T satisfies $F(T) \subset A(T)$ if T is λ-hybrid. So a λ-hybrid mapping T is quasi-nonexpansive if $F(T) \neq \emptyset$. According to Falset et al. [26], T is said to satisfy the condition (E) if there is $s \in [0, \infty)$ such that

$$\|x - Ty\| \le \|x - y\| + s\|x - Tx\|, \tag{6}$$

for all $x, y \in C$. If T satisfies the condition (E), then $F(T) \subset A(T)$. So T is quasi-nonexpansive if T satisfies the condition (E) and $F(T) \neq \emptyset$.

Let C be a subset of H and let T_1 and T_2 be self-mappings on C. We denote by F the common fixed point set $F(T_1) \cap F(T_2)$ and by A the common attractive point set $A(T_1) \cap A(T_2)$. We place importance on the condition $F \subset A$. Therefore, we present some facts relevant to this condition. It is easy to see the following:

(a) $F \subset A$ implies neither $F(T_1) \subset A(T_1)$ nor $F(T_2) \subset A(T_2)$

(b) $F \neq \emptyset$ does not imply $A \neq \emptyset$ without the assumption $F \subset A$

(c) In the case when C is closed and convex, $A \neq \emptyset$ implies $F \neq \emptyset$. However, $A \neq \emptyset$ does not imply $F \subset A$

(d) If T_1 and T_2 are quasi-nonexpansive mappings, then $F \subset A$

(e) Even if $\emptyset \neq F \subset A$, neither T_1 nor T_2 need be quasi-nonexpansive

To clearly understand such situations, a specific example from [20] is given below.

Example 1 (see [20]). Let $D = \{x = (s,t) \in \mathbb{R}^2 : s \in [0,1],\ t \in [s/2, 2s]\}$. Then, D is compact and convex. Let T_1 and T_2 be self-mappings on D defined by

$$T_1 x = T_1(s,t) = \frac{1}{2}\left((s,t) + \left(\frac{1}{2}t, t\right)\right) = \left(\frac{1}{2}s + \frac{1}{4}t, t\right),$$
$$T_2 x = T_2(s,t) = \frac{1}{2}\left((s,t) + \left(s, \frac{1}{2}s\right)\right) = \left(s, \frac{1}{4}s + \frac{1}{2}t\right), \tag{7}$$

for each $x = (s,t) \in D$.

In this example, we easily see

$$\begin{aligned} F(T_1) &= \{(x_1,x_2) \in D : x_2 = 2x_1\}, \\ F(T_2) &= \left\{(x_1,x_2) \in D : x_2 = \frac{x_1}{2}\right\}, \\ A(T_1) &= \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \le 0\right\}, \\ A(T_2) &= \left\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \le 0\right\}, \\ F &= \{(0,0)\}, \\ A &= \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \le 0, x_2 \le 0\right\}. \end{aligned} \tag{8}$$

From this, we can easily confirm the following:

(i) $F(T_1) \cap F(T_2) \subset A(T_1) \cap A(T_2)$ holds

(ii) Neither $F(T_1) \subset A(T_1)$ nor $F(T_2) \subset A(T_2)$ holds

(iii) Neither T_1 nor T_2 is quasi-nonexpansive

Refer to [17–21] for more details of attractive points.

3. Lemmas

We begin this section with preparing the required symbols for the iteration scheme we are dealing with and then present some lemmas which are needed to prove our main result.

Let C be a nonempty subset of a Hilbert space H. Let T_1 and T_2 be self-mappings on C. For each $n \in \mathbb{N}$, define mappings $M_e(n)$, $M_o(n)$, and $M(n)$ from C into H, respectively, by

$$\begin{aligned} M_e(n)x &= \frac{1}{n}\sum_{i=0}^{n-1} T_1^i T_2^i x, \\ M_o(n)x &= \frac{1}{n}\sum_{i=0}^{n-1} T_1^i T_2^{i+1} x, \\ M(n)x &= \frac{1}{2n}\sum_{i=0}^{n-1}\sum_{j=i}^{i+1} T_1^i T_2^j x = \frac{1}{2}M_e(n)x + \frac{1}{2}M_o(n)x, \end{aligned} \tag{9}$$

for each $x \in C$. Then, for each $n \in \mathbb{N}$, $v \in F(M_e(n)) \cap F(M_o(n)) \cap F(M(n))$ holds if $v \in F$, and $u \in A(M_e(n)) \cap A(M_o(n)) \cap A(M(n))$ holds if $u \in A$. In the case of $T_1 T_2 = T_2 T_1$, the following holds:

$$\begin{aligned} T_1^i T_2^i x &= (T_1 T_2)^i x, \\ T_1^i T_2^{i+1} x &= (T_1 T_2)^i (T_2 x) = T_2\left((T_1 T_2)^i x\right), \end{aligned} \tag{10}$$

for all $i \in \mathbb{N}_0$, $x \in C$.

Here are some lemmas which are needed to get our main result. First, since a Hilbert space has the Opial property, the following lemma is easily obtained; for example, see Atsushiba et al. [18].

Lemma 3. *Let H be a Hilbert space and let C be a nonempty subset of H. Let $\{u_n\}$ be a sequence in H such that $\{\|u_n - w\|\}$ converges for each $w \in C$. Suppose $\{u_{n_i}\}$ and $\{u_{n_j}\}$ are subsequences of $\{u_n\}$ which converge weakly to $u, v \in C$, respectively. Then, $u = v$.*

The following lemma is due to Ibaraki and Takeuchi [20].

Lemma 4 (see [20]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a mapping from C into H. Let $a \in [0,1]$, $x \in C$, and $w = ax + (1-a)Tx$. Suppose $v \in A(T)$. Then, the following holds:*

$$a(1-a)\|Tx - x\|^2 \le \|x - v\|^2 - \|w - v\|^2. \tag{11}$$

Suppose further that C is bounded. Let $r > \sup_{x\in C}\|x - v\|$. Then,

$$\frac{a(1-a)}{2r}\|Tx - x\|^2 \le \|x - v\| - \|w - v\|. \tag{12}$$

We need the following trivial lemma to prove Lemma 6.

Lemma 5. *Let $\{a_n\}_{n\in\mathbb{N}_0}$ be a sequence in $[0,\infty)$. Then, for each $n \in \mathbb{N}_2$, there exists $i_n \in \mathbb{N}(0, n-2)$ satisfying $\min_{i\in\mathbb{N}(0,n-2)}(a_i + a_{i+1}) = a_{i_n} + a_{i_n+1}$, and therefore, the following holds:*

$$\sum_{i=0}^{n-1} a_i \ge \frac{n-1}{2}\left(a_{i_n} + a_{i_n+1}\right). \tag{13}$$

Proof. Fix any $n \in \mathbb{N}_2$. It is trivial that there exists $i_n \in \mathbb{N}(0, n-2)$ such that

$$\min_{i \in \mathbb{N}(0,n-2)} (a_i + a_{i+1}) = a_{i_n} + a_{i_n+1}. \tag{14}$$

In the case when n is odd, $n \in \mathbb{N}_3$ and i_n satisfies

$$\begin{aligned}\sum_{i=0}^{n-1} a_i \geq \sum_{i=0}^{n-2} a_i &= \sum_{k=0}^{(n-3)/2} (a_{2k} + a_{2k+1}) \\ &\geq \sum_{k=0}^{(n-3)/2} \left(a_{i_n} + a_{i_n+1}\right) = \frac{n-1}{2}\left(a_{i_n} + a_{i_n+1}\right).\end{aligned} \tag{15}$$

In the case when n is even, i_n satisfies

$$\begin{aligned}\sum_{i=0}^{n-1} a_i &= \sum_{k=0}^{(n-2)/2} (a_{2k} + a_{2k+1}) \geq \sum_{k=0}^{(n-2)/2} \left(a_{i_n} + a_{i_n+1}\right) \\ &= \frac{n}{2}\left(a_{i_n} + a_{i_n+1}\right).\end{aligned} \tag{16}$$

Thus, we see that i_n satisfies (13). □

Lemma 6. *Let H be a Hilbert space and let C be a nonempty subset of H. Let T_1 and T_2 be self-mappings on C satisfying $\emptyset \neq A = A(T_1) \cap A(T_2)$. Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. For each $n \in \mathbb{N}$, let $M(n)$ be as in (9). Let $\{x_n\}$ be a bounded sequence in C and define $\{y_n\}$ by*

$$y_n = a_n M(n) x_n + (1 - a_n) x_n, \tag{17}$$

for each $n \in \mathbb{N}$. Suppose $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$. Then, there is a sequence $\{i_n\}_{n\in\mathbb{N}_2}$ such that $i_n \in \mathbb{N}(0, n-2) \subset \mathbb{N}_0$ for each $n \in \mathbb{N}_2$ and

$$\begin{aligned}\lim_{n\longrightarrow\infty} \left\| T_1^{i_n} T_2^{i_n} x_n - x_n \right\| &= 0, \\ \lim_{n\longrightarrow\infty} \left\| T_1^{i_n} T_2^{i_n+1} x_n - x_n \right\| &= 0, \\ \lim_{n\longrightarrow\infty} \left\| T_1^{i_n+1} T_2^{i_n+1} x_n - x_n \right\| &= 0.\end{aligned} \tag{18}$$

Proof. For each $n \in \mathbb{N}$, let $M_e(n)$ and $M_o(n)$ be as in (9). Note that $0 < a(1-b) \leq a_n(1-a_n)$ for all $n \in \mathbb{N}$. Fix any $u \in A$. Since $\{x_n\}$ is a bounded sequence in C, there is an $r \in (0,\infty)$ satisfying $r > \sup_{n\in\mathbb{N}} \|x_n - u\|$. For each $n \in \mathbb{N}$, by (12) in Lemma 4, we see that

$$\begin{aligned}&\|a_n M_e(n) x_n + (1-a_n) x_n - u\| \\ &= \left\| \frac{a_n}{n} \sum_{i=0}^{n-1} T_1^i T_2^i x_n + (1-a_n) x_n - u \right\| \\ &= \left\| a_n \left(\frac{1}{n} \sum_{i=0}^{n-1} T_1^i T_2^i x_n - u \right) + (1-a_n)(x_n - u) \right\| \\ &= \left\| \frac{1}{n} \sum_{i=0}^{n-1} \left(a_n \left(T_1^i T_2^i x_n - u \right) + (1-a_n)(x_n - u) \right) \right\| \\ &\leq \frac{1}{n} \sum_{i=0}^{n-1} \left\| a_n \left(T_1^i T_2^i x_n - u \right) + (1-a_n)(x_n - u) \right\| \\ &\leq \frac{1}{n} \sum_{i=0}^{n-1} \left(\|x_n - u\| - \frac{a_n(1-a_n)}{2r} \left\| T_1^i T_2^i x_n - x_n \right\|^2 \right) \\ &\leq \|x_n - u\| - \frac{a(1-b)}{2r} \frac{1}{n} \sum_{i=0}^{n-1} \left\| T_1^i T_2^i x_n - x_n \right\|^2.\end{aligned} \tag{19}$$

Then, we have

$$\begin{aligned}&\frac{a(1-b)}{2r} \frac{1}{n} \sum_{i=0}^{n-1} \left\| T_1^i T_2^i x_n - x_n \right\|^2 \\ &\quad \leq \|x_n - u\| - \|a_n M_e(n) x_n + (1-a_n) x_n - u\|.\end{aligned} \tag{20}$$

In the same way, we also have

$$\begin{aligned}&\frac{a(1-b)}{2r} \frac{1}{n} \sum_{i=0}^{n-1} \left\| T_1^i T_2^{i+1} x_n - x_n \right\|^2 \\ &\quad \leq \|x_n - u\| - \|a_n M_o(n) x_n + (1-a_n) x_n - u\|.\end{aligned} \tag{21}$$

By (20) and (21), the following holds:

$$\begin{aligned}&\frac{a(1-b)}{4r} \frac{1}{n} \sum_{i=0}^{n-1} \left(\left\| T_1^i T_2^i x_n - x_n \right\|^2 + \left\| T_1^i T_2^{i+1} x_n - x_n \right\|^2 \right) \\ &\leq \frac{1}{2} \left(\|x_n - u\| - \|a_n M_e(n) x_n + (1-a_n) x_n - u\| \right) \\ &\quad + \frac{1}{2} \left(\|x_n - u\| - \|a_n M_o(n) x_n + (1-a_n) x_n - u\| \right) \\ &\leq \|x_n - u\| - \left\| \frac{1}{2} a_n M_e(n) x_n \right. \\ &\quad \left. + \frac{1}{2} a_n M_o(n) x_n + (1-a_n) x_n - u \right\| \\ &= \|x_n - u\| - \|a_n M(n) x_n + (1-a_n) x_n - u\| \\ &= \|x_n - u\| - \|y_n - u\| \leq \|x_n - y_n\|.\end{aligned} \tag{22}$$

For each $i \in \mathbb{N}_0$, we regard $\|T_1^i T_2^i x_n - x_n\|^2 + \|T_1^i T_2^{i+1} x_n - x_n\|^2$ as a_i in Lemma 5. From Lemma 5, for each $n \in \mathbb{N}_2$, there exists $i_n \in \mathbb{N}(0, n-2) \subset \mathbb{N}_0$ satisfying

(13). So, we have the sequence $\{i_n\}_{n\in\mathbb{N}_2}$ which consists of such i_n. Then, for each $n \in \mathbb{N}_2$, it follows from (13) that

$$\begin{aligned}\frac{n-1}{2}&\left(\left\|T_1^{i_n}T_2^{i_n}x_n - x_n\right\|^2 + \left\|T_1^{i_n}T_2^{i_n+1}x_n - x_n\right\|^2\right.\\&\left.+\left\|T_1^{i_n+1}T_2^{i_n+1}x_n - x_n\right\|^2 + \left\|T_1^{i_n+1}T_2^{i_n+2}x_n - x_n\right\|^2\right)\\&\leq \sum_{i=0}^{n-1}\left(\left\|T_1^iT_2^ix_n - x_n\right\|^2 + \left\|T_1^iT_2^{i+1}x_n - x_n\right\|^2\right).\end{aligned} \tag{23}$$

From this, by (22), we see

$$\begin{aligned}&\frac{a(1-b)}{8r}\frac{n-1}{n}\left\|T_1^{i_n}T_2^{i_n}x_n - x_n\right\|^2\\&\leq \frac{a(1-b)}{4r}\frac{1}{n}\frac{n-1}{2}\left(\left\|T_1^{i_n}T_2^{i_n}x_n - x_n\right\|^2 + \left\|T_1^{i_n}T_2^{i_n+1}x_n - x_n\right\|^2\right.\\&\quad\left.+\left\|T_1^{i_n+1}T_2^{i_n+1}x_n - x_n\right\|^2 + \left\|T_1^{i_n+1}T_2^{i_n+2}x_n - x_n\right\|^2\right)\\&\leq \frac{a(1-b)}{4r}\frac{1}{n}\sum_{i=0}^{n-1}\left(\left\|T_1^iT_2^ix_n - x_n\right\|^2 + \left\|T_1^iT_2^{i+1}x_n - x_n\right\|^2\right)\\&\leq \|x_n - y_n\|.\end{aligned} \tag{24}$$

Then, by $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we immediately see

$$\begin{aligned}&\limsup_{n\to\infty}\frac{a(1-b)}{8r}\frac{n-1}{n}\left\|T_1^{i_n}T_2^{i_n}x_n - x_n\right\|^2\\&\quad\leq \lim_{n\to\infty}\|x_n - y_n\| = 0.\end{aligned} \tag{25}$$

Thus, by $((a(1-b))/8r) > 0$ and $\lim_{n\to\infty}((n-1)/n) = 1$, we obtain

$$\lim_{n\to\infty}\left\|T_1^{i_n}T_2^{i_n}x_n - x_n\right\|^2 = 0. \tag{26}$$

In the same way, we also have the following:

$$\begin{aligned}&\lim_{n\to\infty}\left\|T_1^{i_n}T_2^{i_n+1}x_n - x_n\right\|^2 = 0,\\&\lim_{n\to\infty}\left\|T_1^{i_n+1}T_2^{i_n+1}x_n - x_n\right\|^2 = 0.\end{aligned} \tag{27}$$

□

The following lemmas for a λ-hybrid mapping and a mapping satisfying the condition (E) were known; for example, see Ibaraki and Takeuchi [20]. Of course, these are extensions of the demiclosed principle in the Hilbert space setting.

Lemma 7. *Let H be a Hilbert space and let C be a subset of H. Let T be a λ-hybrid mapping from C into H. Suppose $\{x_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Then, $u \in F(T)$.*

Lemma 8. *Let H be a Hilbert space and let C be a subset of H. Let T be a mapping from C into H which satisfies the condition (E). Suppose $\{x_n\}$ is a sequence in C which converges weakly to some $u \in C$ and satisfies $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Then, $u \in F(T)$.*

4. Main Result and Applications

We present a weak convergence theorem which is our main result.

Theorem 9. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T_1 and T_2 be self-mappings on C satisfying $T_1T_2 = T_2T_1$. Let $a, b \in (0,1)$ satisfy $a \leq b$ and let $\{a_n\}$ be a sequence in $[a,b]$. For each $n \in \mathbb{N}$, let $M(n)$ be as in (9). Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$\begin{aligned}x_{n+1} &= \frac{a_n}{2n}\sum_{i=0}^{n-1}\sum_{j=i}^{i+1}T_1^iT_2^jx_n + (1-a_n)x_n\\&= a_nM(n)x_n + (1-a_n)x_n,\end{aligned} \tag{28}$$

for each $n \in \mathbb{N}$. Set $A = A(T_1) \cap A(T_2)$ and $F = F(T_1) \cap F(T_2)$. Suppose $A \neq \emptyset$ and $I - T_j$ is demiclosed at 0 for each $j \in \{1,2\}$. Then, the following hold:

(1) A weakly convergent subsequence of $\{x_n\}$ exists, and every weakly convergent subsequence of $\{x_n\}$ converges weakly to a point of F

(2) In the case of $F \subset A$, $\{x_n\}$ converges weakly to some $z \in F$

Proof. Fix any $v \in A$. It is trivial that $A \subset A(M(n))$ for all $n \in \mathbb{N}$. Then, by (11) in Lemma 4, we see

$$0 \leq a_n(1-a_n)\|M(n)x_n - x_n\|^2 \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2, \tag{29}$$

for each $n \in \mathbb{N}$. From this, we see that $\|x_{n+1} - v\| \leq \|x_n - v\|$ for all $n \in \mathbb{N}$; that is, $\{\|x_n - v\|\}$ converges. So, since $0 < a(1-b) \leq a_n(1-a_n)$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\|M(n)x_n - x_n\|^2 = 0. \tag{30}$$

Furthermore, we know that for each $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|a_nM(n)x_n + (1-a_n)x_n - x_n\|\\&= a_n\|M(n)x_n - x_n\|.\end{aligned} \tag{31}$$

By (30) and (31), $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ holds. For each $n \in \mathbb{N}$, we may regard x_{n+1} as y_n in Lemma 6. Thus, since $\{x_n\}$ is bounded, by Lemma 6, there is a sequence $\{i_n\}_{n\in\mathbb{N}_2}$ such that $i_n \in \mathbb{N}(0, n-2) \subset \mathbb{N}_0$ for each $n \in \mathbb{N}_2$ and

$$\begin{aligned} \lim_{n\longrightarrow\infty} \left\| T_1^{i_n} T_2^{i_n} x_n - x_n \right\| &= 0, \\ \lim_{n\longrightarrow\infty} \left\| T_1^{i_n} T_2^{i_n+1} x_n - x_n \right\| &= 0, \\ \lim_{n\longrightarrow\infty} \left\| T_1^{i_n+1} T_2^{i_n+1} x_n - x_n \right\| &= 0. \end{aligned} \tag{32}$$

We show that (1) holds. Since $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence. Let $\{x_{n_l}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $z \in H$. Since C is weakly closed and $\{x_{n_l}\}$ is a sequence in C, we see $z \in C$.

We show $z \in F(T_2)$. By $T_1T_2 = T_2T_1$ and (32), we see

$$\begin{aligned} &\lim_{l\longrightarrow\infty} \left\| T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} - x_{n_l} \right\| = 0, \\ &\limsup_{l\longrightarrow\infty} \left\| T_2 \left(T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} \right) - T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} \right\| \\ &\quad \le \limsup_{l\longrightarrow\infty} \left\| T_1^{i_{n_l}} T_2^{i_{n_l+1}} x_{n_l} - x_{n_l} \right\| + \limsup_{l\longrightarrow\infty} \left\| x_{n_l} - T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} \right\| \\ &\quad = \lim_{l\longrightarrow\infty} \left\| T_1^{i_{n_l}} T_2^{i_{n_l+1}} x_{n_l} - x_{n_l} \right\| + \lim_{l\longrightarrow\infty} \left\| x_{n_l} - T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} \right\| = 0. \end{aligned} \tag{33}$$

From the latter, we see $\lim_{l\longrightarrow\infty} \| T_2 (T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l}) - T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} \| = 0$. Since $\{x_{n_l}\}$ converges weakly to $z \in C$, by $\lim_{l\longrightarrow\infty} \| T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l} - x_{n_l} \| = 0$, $\{T_1^{i_{n_l}} T_2^{i_{n_l}} x_{n_l}\}$ also converges weakly to z. So, since $I - T_2$ is demiclosed at 0, we see $z \in F(T_2)$.

We show $z \in F(T_1)$. Similarly to the discussion above, we have the following:

$$\begin{aligned} \lim_{l\longrightarrow\infty} \left\| T_1^{i_{n_l}} T_2^{i_{n_l}+1} x_{n_l} - x_{n_l} \right\| &= 0, \\ \lim_{l\longrightarrow\infty} \left\| T_1 \left(T_1^{i_{n_l}} T_2^{i_{n_l}+1} x_{n_l} \right) - T_1^{i_{n_l}} T_2^{i_{n_l}+1} x_{n_l} \right\| &= 0. \end{aligned} \tag{34}$$

So, $\{T_1^{i_{n_l}} T_2^{i_{n_l}+1} x_{n_l}\}$ converges weakly to z and

$$\lim_{l\longrightarrow\infty} \left\| T_1 \left(T_1^{i_{n_l}} T_2^{i_{n_l}+1} x_{n_l} \right) - T_1^{i_{n_l}} T_2^{i_{n_l}+1} x_{n_l} \right\| = 0. \tag{35}$$

Since $I - T_1$ is demiclosed at 0, we see $z \in F(T_1)$. Thus, $z \in F = F(T_1) \cap F(T_2)$.

By the argument so far, we see that a weakly convergent subsequence of $\{x_n\}$ exists and any weakly convergent subsequence of $\{x_n\}$ has to converge weakly to a point of F. So, we confirmed that (1) holds.

We show that (2) holds. We already know that $\{\|x_n - v\|\}$ converges for each $v \in A$. Suppose $F \subset A$. Then, by (1), $F \subset A$, and Lemma 3, all weak cluster points of $\{x_n\}$ coincide. That is, $\{x_n\}$ itself converges weakly to a point $z \in F$. □

Next, we present some results derived from Theorem 9.

Theorem 10. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T_1 and T_2 be quasi-nonexpansive self-mappings on C satisfying $T_1T_2 = T_2T_1$. Assume that $F = F(T_1) \cap F(T_2) \neq \emptyset$ and $I - T_j$ is demiclosed at 0 for each $j \in \{1, 2\}$. Let $a, b \in (0, 1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a, b]$. For each $n \in \mathbb{N}$, let $M(n)$ be as in (9). Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = a_n M(n) x_n + (1 - a_n) x_n, \tag{36}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since T_1 and T_2 are quasi-nonexpansive, we know that $\emptyset \neq F = F(T_1) \cap F(T_2) \subset A(T_1) \cap A(T_2) = A$ holds. Thus, by Theorem 9, we have the result. □

Theorem 11. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T_1 and T_2 be nonexpansive self-mappings on C satisfying $T_1T_2 = T_2T_1$. Assume $F = F(T_1) \cap F(T_2) \neq \emptyset$. Let $a, b \in (0, 1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a, b]$. For each $n \in \mathbb{N}$, let $M(n)$ be as in (9). Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = a_n M(n) x_n + (1 - a_n) x_n, \tag{37}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to some $z \in F$.

Proof. If a mapping T is nonexpansive and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. We also know that $I - T$ is demiclosed at 0 if T is nonexpansive. Thus, by Theorem 10, we have the result. □

By considering Lemmas 7 and 8, we have the following theorems.

Theorem 12. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T_1 and T_2 be self-mappings on C satisfying $T_1T_2 = T_2T_1$. Assume that T_1 is λ-hybrid, T_2 is μ-hybrid, and $F = F(T_1) \cap F(T_2) \neq \emptyset$. Let $a, b \in (0, 1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a, b]$. For each $n \in \mathbb{N}$, let $M(n)$ be as in (9). Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = a_n M(n) x_n + (1 - a_n) x_n, \tag{38}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since T_1 is λ-hybrid and T_2 is μ-hybrid, by $F = F(T_1) \cap F(T_2) \neq \emptyset$, we know that T_1 and T_2 are quasi-nonexpansive. By Lemma 7, we also know that $I - T_1$ and $I - T_2$ are demiclosed at 0. Thus, by Theorem 10, we have the result. □

Theorem 13. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T_1 and T_2 be self-mappings on C satisfying $T_1T_2 = T_2T_1$. Assume that T_1 and T_2 satisfy the condition (E) and $F = F(T_1) \cap F(T_2) \neq \emptyset$.*

Let $a, b \in (0,1)$ satisfy $a \le b$ and let $\{a_n\}$ be a sequence in $[a, b]$. For each $n \in \mathbb{N}$, let $M(n)$ be as in (9). Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by

$$x_{n+1} = a_n M(n) x_n + (1 - a_n) x_n, \tag{39}$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to some $z \in F$.

Proof. Since T_1 and T_2 satisfy the condition (E), by $F = F(T_1) \cap F(T_2) \neq \varnothing$, we know that T_1 and T_2 are quasi-nonexpansive. By Lemma 8, we also know that $I - T_1$ and $I - T_2$ are demiclosed at 0. Thus, by Theorem 10, we have the result. □

5. Supplement

In this section, we present some examples that complement the argument so far. In advance, recall that weak and strong topologies on a Euclidean space coincide.

In the previous section, Theorem 9, neither T_1 nor T_2 need to be quasi-nonexpansive. However, in Theorems 10–13, we deal only with quasi-nonexpansive mappings T_1 and T_2. We therefore give an example where all conditions of Theorem 9 are satisfied and T_1 is not quasi-nonexpansive.

Example 2. Let $D = \{x = (s, t) \in \mathbb{R}^2 : s \in [0, 1],\ t \in [-s, s]\}$. Let T_1 and T_2 be continuous self-mappings on D defined, respectively, by

$$\begin{aligned} T_1 x &= T_1(s,t) = \left(\frac{1}{2}(s + |t|), t\right), \\ T_2 x &= T_2(s,t) = (s, -t), \end{aligned} \tag{40}$$

for each $x = (s, t) \in D$.

In this example, noting $F = F(T_1) \cap F(T_2)$ and $A = A(T_1) \cap A(T_2)$, we see

$$\begin{aligned} F(T_1) &= \{(x_1, x_2) \in D : x_1 = |x_2|\}, \\ F(T_2) &= \{(x_1, x_2) \in D : x_2 = 0\}, \\ A(T_1) &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0\}, \\ A(T_2) &= \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}, \\ F &= \{(0,0)\}, \\ A &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le 0, x_2 = 0\}. \end{aligned} \tag{41}$$

From these, we can easily verify the following:

(i) D is a nonempty compact convex subset of $\mathbb{R}^2$

(ii) $F(T_1) \not\subset A(T_1)$, that is, T_1 is not quasi-nonexpansive

(iii) T_2 is nonexpansive and $F(T_2) \neq \varnothing$; that is, T_2 is quasi-nonexpansive

(iv) $I - T_1$ and $I - T_2$ are demiclosed at 0, since T_1 and T_2 are continuous

(v) $T_1 T_2 = T_2 T_1$ and $\varnothing \neq F \subset A$

So, all conditions of Theorem 9 are satisfied and T_1 is not quasi-nonexpansive. This implies that the sequence $\{y_n\}$ generated by the procedure in Theorem 9 converges strongly to $(0,0) \in F \subset A$.

Recall that the condition $F \subset A$ is unnecessary to prove (1) of Theorem 9. That is, without the condition $F \subset A$, we proved that the sequence $\{x_n\}$ in Theorem 9 has a subsequence which converges weakly to a point $z \in F$. So, to gain a better understanding the contents of this paper, we are interested in what happens when the condition $F \subset A$ is missing from Theorem 9. Then, from this point of view, we give the following example.

Example 3. Let $D = [0, 1]$. Let T_1 and T_2 be continuous self-mappings on D defined, respectively, by

$$\begin{aligned} T_1 s &= 2s^2 \text{ when } s \in \left[0, \frac{1}{2}\right], \\ T_1 s &= 2\left(s - \frac{1}{2}\right)^2 + \frac{1}{2} \text{ when } s \in \left(\frac{1}{2}, 1\right], \\ T_2 s &= 4s^3 \text{ when } s \in \left[0, \frac{1}{2}\right], \\ T_2 s &= 4\left(s - \frac{1}{2}\right)^3 + \frac{1}{2} \text{ when } s \in \left(\frac{1}{2}, 1\right]. \end{aligned} \tag{42}$$

For this example, we easily see

$$\begin{aligned} F(T_1) &= F(T_2) = \left\{0, \frac{1}{2}, 1\right\}, \\ A(T_1) &= A(T_2) = (-\infty, 0], \\ F &= F(T_1) \cap F(T_2) = \left\{0, \frac{1}{2}, 1\right\}, \\ A &= A(T_1) \cap A(T_2) = (-\infty, 0], \\ (T_1 T_2) s &= 2\left(4s^3\right)^2 = 32 s^6 = 4\left(2s^2\right)^3 \\ &= (T_2 T_1) s \text{ when } s \in \left[0, \frac{1}{2}\right], \\ (T_1 T_2) s &= 2\left(4\left(s - \frac{1}{2}\right)^3 + \frac{1}{2} - \frac{1}{2}\right)^2 + \frac{1}{2} \\ &= 32\left(s - \frac{1}{2}\right)^6 + \frac{1}{2}, \\ (T_2 T_1) s &= 4\left(2\left(s - \frac{1}{2}\right)^2 + \frac{1}{2} - \frac{1}{2}\right)^3 + \frac{1}{2} \\ &= 32\left(s - \frac{1}{2}\right)^6 + \frac{1}{2} \text{ when } s \in \left(\frac{1}{2}, 1\right]. \end{aligned} \tag{43}$$

From these, we see the following:

(i) D is a nonempty compact convex subset of $\mathbb{R}$

(ii) Neither T_1 nor T_2 is quasi–nonexpansive

(iii) $I - T_1$ and $I - T_2$ are demiclosed at 0, since T_1 and T_2 are continuous

(iv) $T_1 T_2 = T_2 T_1$ and $A \neq \varnothing$

So, without $F \subset A$, all conditions of Theorem 9 are satisfied.

In this example, for ease of verification, we have chosen T_1 and T_2 which are special in the following sense:

(i) We can regard T_1 and T_2 as self-mappings on [0, 1/2]

(ii) We can regard T_1 and T_2 as self-mappings on [1/2, 1]

Accordingly, the following should be noted:

(i) $F = \{0, 1/2\}$ and $A = (-\infty, 0]$ if T_1 and T_2 are considered like the former

(ii) $F = \{1/2, 1\}$ and $A = (-\infty, 1/2]$ if T_1 and T_2 are considered like the latter

We know that the sequence $\{x_n\}$ generated by the procedure in Theorem 9 has a subsequence which converges strongly to a point z of F. Especially, in this example, we can confirm that the sequence $\{x_n\}$ itself converges strongly to z. That is, by the argument so far, we can easily verify the following:

(i) $\{x_n\}$ converges strongly to $0 \in F \cap A$ if $x_1 \in [0, 1/2)$

(ii) $\{x_n\}$ converges strongly to $1/2 \in F$ if $x_1 \in [1/2, 1)$

(iii) $\{x_n\}$ converges strongly to $1 \in F$ if $x_1 = 1$

From this, we simultaneously see the following: Example 3 is illustrative example such that $z \in A$ may not hold when the condition $F \subset A$ is missing.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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