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Blow-up result for a plate equation with fractional damping and nonlinear source terms

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Abstract: In this work, we consider a plate equation with nonlinear source and partially hinged boundary conditions. Our goal is to show analytically that the solution blows up in finite time. The background of the problem comes from the modeling of the downward displacement of suspension bridge using a thin rectangular plate. The result in the article shows that in the present of fractional damping and a nonlinear source such as the earthquake shocks, the suspension bridge is bound to collapse in finite time.

Keywords: Blow up, fractional damping, plate equation, fourth-order, partially hinged.

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1. Introduction

In this paper, we consider the following problem

$$\begin{cases} u_{tt} + \mu\partial_t^{1+\alpha}u + \Delta^2 u + a(x,y,t)u = |u|^{p-1}u, \text{ in } \Omega \times (0,T), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \text{ in } \Omega \end{cases} \tag{1}$$

with partially hinged boundary condition

$$\begin{cases} u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for } (y,t) \in (-\ell,\ell) \times (0,T), \\ u(L,y,t) = u_{xx}(L,y,t) = 0, & \text{for } (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \nu u_{xx}(x,\pm\ell,t) = 0, & \text{for } (x,t) \in (0,L) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\nu)u_{xxy}(x,\pm\ell,t) = 0, & \text{for } (x,t) \in (0,L) \times (0,T), \end{cases} \tag{2}$$

where $\Omega = (0,L) \times (-\ell,\ell) \subset \mathbb{R}^2$ represent a thin rectangular plate as a model of a suspension bridge and $u = u(x,y,t)$ is the downward displacement of the rectangular plate, see [1,2] for detail description of suspension bridge models. The function $a = a(x,y,t)$ is bounded, continuous and sign changing. For instance, if $h : [0,\infty) \to (-\infty,\infty)$ be any function and $g : \Omega \to (-\infty,\infty)$ be a bounded function, then $a(x,t) = (signh)(t)g(x)$ is example of a sign changing function. Furthermore, $\mu > 0, 0 < \nu < \frac{1}{2}, 1 < p < \infty$ and $-1 < \alpha < 1$. The notation $\partial_t^{1+\alpha}$ stand for the Capito's fractional derivative (see [3,4]) of order $1+\alpha$ with respect to t defined by

$$\partial_t^{1+\alpha}u(t) = \begin{cases} I^{-\alpha}\frac{du(t)}{dt}, & \text{if } -1 < \alpha < 0, \\ \\ I^{1-\alpha}\frac{d^2u(t)}{dt^2}, & \text{if } 0 < \alpha < 1, \end{cases} \tag{3}$$

where $I^\beta(\beta > 0)$ is the fractional derivative defined by

$$I^\beta \frac{du(t)}{dt} = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}u(\tau)d\tau. \tag{4}$$

For $-1 < \alpha < 0$, the term $\partial_t^{1+\alpha} u$ is called the fractional damping while for $\alpha = -1$ and $\alpha = 0$, it represent respectively the weak and strong damping. We should mention here that the fractional damping plays a dissipative role that is sandwich between the weak and the strong damping (see [5]). Concerning blow up results for plate equations, we mention among others the result of Messaoudi [6], where he studied the Petrovsky equation

$$u_{tt} + \Delta^2 u + a|u_t|^{m-2} u_t = b|u|^{p-2} u, \tag{5}$$

where $a, b > 0$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. He established local existence and uniqueness of a weak local solution and that for negative initial energy $(E(0) < 0)$ the local solution blows up in finite time when $p > m$. In addition, established the existence of global solution when $m \geq p$. The result in [6] was later improved by Chen and Zhou in [7]. Li *et al.* [8] considered

$$u_{tt} + \Delta^2 u - \Delta u + |u_t|^{m-1} u_t = |u|^{p-1} u \tag{6}$$

and established global existence and blow up of solutions. Piskin and Polat [9] considered (6) and investigated the decay of solutions. Alaimia and Tatar [10] studied

$$\begin{cases} u_{tt} - \Delta u + \partial_t^{1+\alpha} u = |u|^{p-1} u, x \in \Omega, t > 0 \\ u = 0, \partial\Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega \end{cases} \tag{7}$$

and proved blow up of the solutions for negative initial energy. For related results with fractional damping, we refer the reader to [11–18] and references therein. The article is organized as follows: In Section 2, we recall some fundamental materials and useful assumptions on the relaxation function g. In Section 3, we state and prove some technical lemmas. Finally, in section 4, we establish a blow-up result for problem 1.

2. Preliminaries

Throughout the paper, C_i, $i = 1, 2, 3, ..$ or c are generic positive constants that may change within lines and $(,)_2$ and $\|.\|_2$ denote respectively the inner product and norm in $L^2(\Omega)$. We recall some useful materials. We consider the Hilbert space (see [1])

$$H_*^2(\Omega) = \left\{ w \in H^2(\Omega) : w = 0 \text{ on } \{0, L\} \times (-\ell, \ell) \right\},$$

together with the inner product

$$(u, v)_{H_*^2} = \int_\Omega [(\Delta u \Delta v + (1-\nu)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})] dxdy,$$

and denote by $\mathcal{H}(\Omega)$ the dual of $H_*^2(\Omega)$.

Lemma 1. *(Embedding, see [19]) Suppose* $1 < p < +\infty$. *Then for any* $u \in H_*^2(\Omega)$, *there exists an embedding constant* $S_p = S_p(\Omega, p) > 0$ *such that*

$$\|u\|_{L^q(\Omega)} \leq S_p \|u\|_{H_*^2(\Omega)}, \tag{8}$$

where $S_p = \left(\frac{L}{2\ell} + \frac{\sqrt{2}}{2}\right) (2L\ell)^{\frac{p+2}{2p}} \left(\frac{1}{1-\nu}\right)^{\frac{1}{2}}$.

The eigenvalue problem

$$\begin{cases} \Delta^2 u = \lambda u, \quad (x,y) \in \Omega, \\ u(0,y) = u_{xx}(0,y) = u(L,y) = u_{xx}(L,y) = 0, for\ y \in (-\ell, \ell), \\ u_{yy}(x, \pm\ell) + \nu u_{xx}(x, \pm\ell) = 0, \ for\ x \in \ (0, L), \\ u_{yyy}(x, \pm\ell) + (2-\nu) u_{xxy}(x, \pm\ell) = 0, \ for\ x \in \ (0, L) \end{cases} \tag{9}$$

which has been studied in[1], has a unique eigenvalue $\wedge_1 \in (1-\nu, 1),\ 0 < \nu < \frac{1}{2}$ and $\lambda = \wedge_1^2$ is the least eigenvalue. As a consequent, we have the following lemma

Lemma 2. *Suppose* $-\wedge_1 < a_1 \leq a \leq a_2$. *Then, the following inequality holds*

$$A_1 \|u\|^2_{H^2_*(\Omega)} \leq \|u\|^2_{H^2_*(\Omega)} + (au, u)_2 \leq A_2 \|u\|^2_{H^2_*(\Omega)}, \tag{10}$$

where $A_1 = \begin{cases} 1 + \frac{a_1}{\wedge_1}, & a_1 < 0, \\ 1, & a_1 \geq 0 \end{cases}$ *and* $A_2 = \begin{cases} 1, & a_2 < 0, \\ 1 + \frac{a_2}{\wedge_1}, & a_2 \geq 0 \end{cases}$ *which has been proved in [19].*

For completeness, we state without proof a local existence result for problem (1)-(2) (see [19,20] for more on existence).

Theorem 1. *Let* $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$ *be given and assume* $-\wedge_1 < a_1 \leq a \leq a_2$. *Then, there exists a weak unique local solution to problem* $(1)-(2)$ *in the class*

$$u \in L^\infty\left([0,T), H^2_*(\Omega)\right), u_t \in L^\infty\left([0,T), L^2(\Omega)\right), u_{tt} \in L^\infty([0,T), \mathcal{H}(\Omega)), \tag{11}$$

for some $T > 0$.

Definition 1. A function u satisfying (11) is called a weak solution of (1) if

$$\frac{d}{dt}(u_t(t), w)_2 + \frac{\mu}{\Gamma(-\alpha)} \int_\Omega w \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy + (u(t), w)_{H^2_*(\Omega)} + (au(t), w)_2 = \int_\Omega |u|^{p-1} w \, dx dy \tag{12}$$

a.e $t \in (0,T)$ and $\forall w \in H^2_*(\Omega)$.

We consider the energy functional $E(t)$ defined by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2}(au(t), u(t))_2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx dy. \tag{13}$$

Multiplying (1) by u_t and integrating over Ω, using integration by part, definition of fractional derivative (4) and recalling that $a_1 \leq a \leq a_2$, we obtain

$$E'(t) = -\frac{\mu}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy \tag{14}$$

for almost all $t \in [0,T)$. The result in (14) is for any regular solutions. However, this result remains valid for weak solutions by simple density argument. We define a modify energy functional:

$$E_\epsilon(t) = E(t) - \epsilon(u, u_t)_2, \tag{15}$$

for some ϵ to be specified later. Differentiating (15) and making use of $(1)_1$ and (14), we arrive at

$$\begin{aligned} E'_\epsilon(t) = & -\frac{\epsilon\mu}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy - \epsilon\|u_t(t)\|_2^2 + \epsilon\|u(t)\|^2_{H^2_*(\Omega)} \\ & + \frac{\epsilon\mu}{\Gamma(-\alpha)} \int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy + \epsilon(au, u)_2 - \epsilon \int_\Omega |u|^{p+1} dx dy \,. \end{aligned} \tag{16}$$

Also, we define the functional

$$H(t) = -\left(e^{-\gamma\epsilon t} E_\epsilon(t) + \theta F(t) + \lambda\right), \tag{17}$$

where

$$F(t) = \int_\Omega \int_0^t M(t-s) e^{-\gamma\epsilon s} u_s^2(x,y,s) ds dx dy \tag{18}$$

with
$$M(t) = e^{\beta t}\int_t^{+\infty} e^{-\beta s} s^{-(\alpha+1)} ds, \tag{19}$$
where $\gamma = \frac{p+1}{2}$ and θ, λ, β are positive constants to be specified later. The differentiation of (18) gives the relation
$$F'(t) = \beta^\alpha \Gamma(-\alpha) e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \beta F(t) - \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) ds dx dy. \tag{20}$$

In the next section, we state and prove some useful Lemmas.

3. Technical lemma

Lemma 3. *Suppose $E_\epsilon(0) < 0$ and p is sufficiently large, then $H(t)$ and $H'(t)$ are strictly positive.*

Proof. Differentiating (17) with respect to t and using (15) yields
$$\begin{aligned} H'(t) &= \gamma\epsilon e^{-\gamma\epsilon t} E_\epsilon(t) - e^{-\gamma\epsilon t} E'_\epsilon(t) - \theta F'(t) \\ &= \gamma\epsilon e^{-\gamma\epsilon t} E(t) - \gamma\epsilon^2 e^{-\gamma\epsilon t}(u, u_t)_2 - e^{-\gamma\epsilon t} E'_\epsilon(t) - \theta F'(t). \end{aligned} \tag{21}$$

Substituting (13),(16) and (20) into (21), we arrive at
$$\begin{aligned} H'(t) = &\left[\frac{\gamma\epsilon}{2} + \epsilon - \beta^\alpha \theta \Gamma(-\alpha)\right] e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \left[\frac{\gamma\epsilon}{2} - \epsilon\right] e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\ &+ \left[\frac{\gamma\epsilon}{2} - \epsilon\right] e^{-\gamma\epsilon t}(au, u)_2 + \left[\epsilon - \frac{\gamma\epsilon}{p+1}\right] e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1} dx dy \\ &- \gamma\epsilon^2 e^{-\gamma\epsilon t}(u, u_t)_2 + \frac{\mu e^{-\gamma\epsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy \\ &- \frac{\epsilon\mu e^{-\gamma\epsilon t}}{\Gamma(-\alpha)}\int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy \\ &+ \theta \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) ds dx dy - \beta\theta F(t). \end{aligned} \tag{22}$$

Using Young's inequality and Lemma 1, we obtain
$$(u, u_t)_2 \leq \delta_1 S_2^2 \|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{4\delta_1}\|u_t(t)\|_2^2, \quad \delta_1 > 0. \tag{23}$$

Again, Young's and Cauchy-Schwarz inequalities, we get
$$\begin{aligned} &e^{-\gamma\epsilon t}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy \\ &\leq \delta_2 e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \frac{e^{-\gamma\epsilon t}}{4\delta_2}\int_\Omega \left(\int_0^t (t-s)^{-\frac{(\alpha+1)}{2} - \frac{(\alpha+1)}{2}} u_s(s) ds\right)^2 dx dy \\ &\leq \delta_2 e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \frac{(\gamma\epsilon)^\alpha \Gamma(-\alpha)}{4\delta_2}\int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) ds dx dy, \ \ \delta_2 > 0. \end{aligned} \tag{24}$$

In a similar way, with the help of lemma 1, we find
$$\begin{aligned} &e^{-\gamma\epsilon t}\int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy \\ &\leq \delta_3 S_2^2 e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{(\gamma\epsilon)^\alpha \Gamma(-\alpha)}{4\delta_3}\int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) ds dx dy, \ \delta_3 > 0. \end{aligned} \tag{25}$$

Substitution of (23)-(25) into (22) and using lemma 2, we obtain

$$
\begin{aligned}
H'(t) \geq & \left[\frac{\gamma\epsilon}{2} + \epsilon - \beta^{\alpha}\theta\Gamma(-\alpha) - \frac{\gamma\epsilon^2}{4\delta_1} - \frac{\delta_2}{\Gamma(-\alpha)}\right] e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 \\
& + \left[\frac{A_1\gamma\epsilon}{2} - A_1\epsilon - \delta_1 S_2^2\gamma\epsilon^2 - \frac{\delta_3 S_2^2\epsilon\mu}{\Gamma(-\alpha)}\right] e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\
& + \left[\epsilon - \frac{\gamma\epsilon}{p+1}\right] e^{-\gamma\epsilon t}\int_{\Omega}|u|^{p+1}dxdy - \beta\theta F(t) \\
& + \left[\theta - \frac{\mu(\gamma\epsilon)^{\alpha}}{4\delta_2} - \frac{\mu\epsilon(\gamma\epsilon)^{\alpha}}{4\delta_3}\right]\int_{\Omega}\int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma\epsilon s}u_s^2(s)dsdxdy.
\end{aligned} \tag{26}
$$

Adding $C_1H(t) - C_1H(t)$ to the right hand of (26), for some C_1 to be precise, we arrive

$$
\begin{aligned}
H'(t) \geq & \; C_1H(t) + \left[\frac{C_1}{2} + \frac{\gamma\epsilon}{2} + \epsilon - \beta^{\alpha}\theta\Gamma(-\alpha) - \frac{\gamma\epsilon^2}{4\delta_1} - \frac{\delta_2}{\Gamma(-\alpha)}\right] e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 \\
& + \left[\frac{A_1C_1}{2} + \frac{A_1\gamma\epsilon}{2} - A_1\epsilon - \delta_1 S_2^2\gamma\epsilon^2 - \frac{\delta_3 S_2^2\epsilon\mu}{\Gamma(-\alpha)}\right] e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\
& - C_1\epsilon e^{-\gamma\epsilon t}(u,u_t)_2 + \left[\epsilon - \frac{\gamma\epsilon}{p+1} - \frac{C_1}{p+1}\right] e^{-\gamma\epsilon t}\int_{\Omega}|u|^{p+1}dxdy \\
& + \left[\theta - \frac{\mu(\gamma\epsilon)^{\alpha}}{4\delta_2} - \frac{\mu\epsilon(\gamma\epsilon)^{\alpha}}{4\delta_3}\right]\int_{\Omega}\int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma\epsilon s}u_s^2(s)dsdxdy \\
& + (C_1 - \beta)\theta F(t) + C_1\lambda.
\end{aligned} \tag{27}
$$

Applying (23) to (27), we arrive at

$$
\begin{aligned}
H'(t) \geq & \left[\frac{C_1}{2} + \frac{\gamma\epsilon}{2} + \frac{\epsilon}{2} - \beta^{\alpha}\theta\Gamma(-\alpha) - \frac{\gamma\epsilon^2}{4\delta_1} - \frac{\delta_2}{\Gamma(-\alpha)} - \frac{C_1\epsilon}{4\delta_1}\right] e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 \\
& + \left[\frac{A_1C_1}{2} + \frac{A_1\gamma\epsilon}{2} - A_1\epsilon - \delta_1 S_2^2\gamma\epsilon^2 - C_1\delta_1 S_2^2\epsilon - \frac{\delta_3 S_2^2\epsilon\mu}{\Gamma(-\alpha)}\right] e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\
& + \left[\epsilon - \frac{\gamma\epsilon}{p+1} - \frac{C_1}{p+1}\right] e^{-\gamma\epsilon t}\int_{\Omega}|u|^{p+1}dxdy + C_1H(t) + (C_1 - \beta)\theta F(t) + C_1\lambda \\
& + \left[\theta - \frac{\mu(\gamma\epsilon)^{\alpha}}{4\delta_2} - \frac{\mu\epsilon(\gamma\epsilon)^{\alpha}}{4\delta_3}\right]\int_{\Omega}\int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma\epsilon s}u_s^2(s)dsdxdy.
\end{aligned} \tag{28}
$$

Recalling that $\gamma = \frac{p+1}{2}$ and choosing $\delta_1 = \frac{1}{2}, \delta_2 = \delta_3 = \frac{\Gamma(-\alpha)\epsilon}{2}$ and $C_1 = \frac{(p+1)\epsilon}{2}$, we get

$$
\begin{aligned}
H'(t) \geq & \frac{(p+1)\epsilon}{2}H(t) + \left[\frac{p+1}{2}\epsilon(1-\epsilon) - \beta^{\alpha}\theta\Gamma(-\alpha)\right] e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 \\
& + \frac{\epsilon}{2}\left[A_1(p-1) - \epsilon S_2^2((p+1)+\mu)\right] e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\
& + \left(\frac{(p+1)\epsilon}{2} - \beta\right)\theta F(t) + \frac{(p+1)\epsilon}{2}\lambda \\
& + \left[\theta - \frac{\mu(p+1)^{\alpha}\epsilon^{\alpha-1}}{2^{\alpha+1}\Gamma(-\alpha)}(1+\epsilon)\right]\int_{\Omega}\int_0^t (t-s)^{-(\alpha+1)}e^{-\gamma\epsilon s}u_s^2(s)dsdxdy.
\end{aligned} \tag{29}
$$

Now, choosing

$$
\epsilon < \epsilon_1 := \min\left\{1, \frac{A_1(p-1)}{2S_2^2((p+1)+\mu)}\right\}, \tag{30}
$$

we get that

$$
\frac{\epsilon}{2}\left[A_1(p-1) - \epsilon S_2^2((p+1)+\mu)\right] > \frac{A_1(p-1)\epsilon}{4}.
$$

Next, we select $\beta = 1$, we see that for sufficiently large values of p

$$\frac{(p+1)\epsilon}{2} - \beta > 0.$$

Finally, we choose θ such that the coefficient of the second term is non-negative and the coefficient of the last term is greater than $\frac{\mu(p+1)^\alpha}{2^{\alpha+1}\epsilon^{1-\alpha}\Gamma(-\alpha)}$. Thus, we arrive at

$$\begin{aligned} H'(t) \geq \frac{(p+1)\epsilon}{2} H(t) + \frac{A_1(p-1)\epsilon}{4} e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\ + \frac{\mu(p+1)^\alpha}{2^{\alpha+1}\epsilon^{1-\alpha}\Gamma(-\alpha)} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) ds dx dy. \end{aligned} \tag{31}$$

If we choose $\lambda < -E_\epsilon(0)$, then $H(0) > 0$. Consequently, it follows from (31) that $H(t) > 0$ and $H'(t) > 0$. This completes the proof. □

4. Main results

In this section, we show that the solutions of 1-2 blows up in finite time for negative initial energy.

Theorem 2. *Assume that* $-\wedge_1 < a_1 \leq a \leq a_2,\ -1 < \alpha < 0,\ E(0) < 0$ *and* $(u_0, u_1)_2 \geq 0$. *Then the solutions of* 1-2 *blows up in finite time for sufficiently large values of* p.

Proof. We begin by defining the functional G by

$$G(t) = H^{1-\sigma}(t) + \eta e^{-\gamma\epsilon t}(u, u_t)_2, \tag{32}$$

where $\sigma = \frac{p-1}{2(p+1)}$ and $\eta > 0$ to be specified later. Then differentiating $G(t)$ and using (1) yields

$$\begin{aligned} G'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) - \eta\gamma\epsilon e^{-\gamma\epsilon t}(u, u_t)_2 + \eta e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \eta e^{-\gamma\epsilon t}(u, u_{tt})_2 \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) - \eta\gamma\epsilon e^{-\gamma\epsilon t}(u, u_t)_2 + \eta e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \eta e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1} dx dy \\ &\quad - \eta e^{-\gamma\epsilon t}(au, u)_2 - \eta e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} - \frac{\mu\eta e^{-\gamma\epsilon t}}{\Gamma(-\alpha)} \int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) dx dy. \end{aligned} \tag{33}$$

Similarly as in the inequalities (23)and (25), we have that

$$(u, u_t)_2 \leq \delta_4 S_2^2 \|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{4\delta_4}\|u_t(t)\|_2^2, \quad \delta_4 > 0. \tag{34}$$

and

$$\begin{aligned} & e^{-\gamma\epsilon t}\int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_s(s) ds dx dy \\ & \leq \delta_5 e^{-\gamma\epsilon t}\|u(t)\|_2^2 + \frac{(\gamma\epsilon)^\alpha \Gamma(-\alpha)}{4\delta_5} \int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) ds dx dy,\ \delta_5 > 0. \end{aligned} \tag{35}$$

From Lemma 2, we get

$$A_1 e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \leq e^{-\gamma\epsilon t}\left(\|u(t)\|^2_{H^2_*(\Omega)} + (au, u)_2\right). \tag{36}$$

Substituting (34)-(36) into (33), we obtain

$$\begin{aligned} G'(t) \quad \geq \quad & (1-\sigma)H^{-\sigma}(t)H'(t) + \eta\left(1 - \frac{\gamma\epsilon}{4\delta_4}\right) e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 - \eta\left(A_1 + \delta_4\gamma\epsilon S_2^2\right) e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\ & + \eta e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1} dx dy - \frac{\eta\mu\delta_5}{\Gamma(-\alpha)} e^{-\gamma\epsilon t}\|u(t)\|_2^2 - \frac{\mu\eta(\gamma\epsilon)^\alpha}{4\delta_5}\int_\Omega \int_0^t (t-s)^{-(\alpha+1)} e^{-\gamma\epsilon s} u_s^2(s) dx dy \end{aligned} \tag{37}$$

Using (31), we obtain

$$\begin{aligned} G'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \eta\left(1-\frac{\gamma\epsilon}{4\delta_4}\right)e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 \\ &- \eta\left(A_1+\delta_4\gamma\epsilon S_2^2\right)e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} + \eta e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1}dxdy - \frac{\eta\mu\delta_5}{\Gamma(-\alpha)}e^{-\gamma\epsilon t}\|u(t)\|_2^2 \\ &+ \frac{2^{\alpha-1}\gamma^\alpha\eta\epsilon\Gamma(-\alpha)}{\delta_5(p+1)^\alpha}\left(-H'(t)+\frac{p+1}{2}\epsilon H(t)+\frac{A_1(p-1)\epsilon}{4}e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)}\right). \end{aligned} \tag{38}$$

From the last inequality, we get

$$\begin{aligned} G'(t) &\geq \left[(1-\sigma)H^{-\sigma}(t) - \frac{\eta 2^{\alpha-1}\gamma^\alpha\epsilon\Gamma(-\alpha)}{\delta_5(p+1)^\alpha}\right]H'(t) + \frac{\eta 2^{\alpha-2}\gamma^\alpha\epsilon^2\Gamma(-\alpha)}{\delta_5(p+1)^{\alpha-1}}H(t) \\ &- \eta\left[A_1+\delta_4\gamma\epsilon S_2^2 - \frac{2^{\alpha-3}\gamma^\alpha A_1(p-1)\epsilon^2\Gamma(-\alpha)}{\delta_5(p+1)^\alpha}\right]e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\ &+ \eta\left(1-\frac{\gamma\epsilon}{4\delta_4}\right)e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \eta e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1}dxdy - \frac{\eta\mu\delta_5}{\Gamma(-\alpha)}e^{-\gamma\epsilon t}\|u(t)\|_2^2. \end{aligned} \tag{39}$$

Now, we choose $\delta_5 = BH^\sigma(t)$ for some B positive to be precise later. Then, (39) becomes

$$\begin{aligned} G'(t) &\geq \left[(1-\sigma) - \frac{\eta 2^{\alpha-1}\gamma^\alpha\epsilon\Gamma(-\alpha)}{B(p+1)^\alpha}\right]H^{-\sigma}(t)H'(t) + \frac{\eta 2^{\alpha-2}\gamma^\alpha\epsilon^2\Gamma(-\alpha)}{B(p+1)^{\alpha-1}}H^{1-\sigma}(t) \\ &- \eta\left[A_1+\delta_4\gamma\epsilon S_2^2 - \frac{2^{\alpha-3}\gamma^\alpha A_1(p-1)\epsilon^2\Gamma(-\alpha)H^{-\sigma}(t)}{B(p+1)^\alpha}\right]e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\ &+ \eta\left(1-\frac{\gamma\epsilon}{4\delta_4}\right)e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \eta e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1}dxdy - \frac{\eta\mu B}{\Gamma(-\alpha)}e^{-\gamma\epsilon t}H^\sigma(t)\|u(t)\|_2^2. \end{aligned} \tag{40}$$

Adding and subtracting $H(t)$ on the right hand side of (40) and making use of lemma 2 leads to

$$\begin{aligned} G'(t) &\geq \left[(1-\sigma) - \frac{\eta 2^{\alpha-1}\gamma^\alpha\epsilon\Gamma(-\alpha)}{B(p+1)^\alpha}\right]H^{-\sigma}(t)H'(t) + \left[1+\frac{\eta 2^{\alpha-2}\gamma^\alpha\epsilon^2\Gamma(-\alpha)}{B(p+1)^{\alpha-1}}H^{-\sigma}(t)\right]H(t) \\ &+ \left[\frac{A_1}{2} + \frac{\eta 2^{\alpha-3}\gamma^\alpha A_1(p-1)\epsilon^2\Gamma(-\alpha)}{B(p+1)^\alpha}H^{-\sigma}(t) - \eta(A_1+\delta_4\gamma\epsilon S_2^2)\right]e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \\ &+ \left[\frac{1}{2}+\eta\left(1-\frac{\gamma\epsilon}{4\delta_4}\right)\right]e^{-\gamma\epsilon t}\|u_t(t)\|_2^2 + \left[\eta - \frac{1}{p+1}\right]e^{-\gamma\epsilon t}\int_\Omega |u|^{p+1}dxdy \\ &- \epsilon e^{-\gamma\epsilon t}(u,u_t)_2 + \theta F(t) + \lambda - \frac{\eta\mu B}{\Gamma(-\alpha)}e^{-\gamma\epsilon t}H^\sigma(t)\|u(t)\|_2^2. \end{aligned} \tag{41}$$

The term $(u,u_t)_2$ is estimated similarly as in (23) as

$$(u,u_t)_2 \leq \delta_6 S_2^2\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{4\delta_6}\|u_t(t)\|_2^2, \quad \delta_6 > 0. \tag{42}$$

For the term $H^\sigma(t)\|u(t)\|_2^2$, we use the definition of $H(t)$ in (17) and the choice of ϵ in (30) to get (see [10] page 141 for detail computations)

$$H^\sigma(t)\|u(t)\|_2^2 \leq \frac{C_2}{(p+1)^\sigma}\left(1+\int_\Omega |u|^{p+1}dxdy\right) \tag{43}$$

for some constant $C_2 > 0$. Substituting (42) and (43) into (41) yields

$$\begin{aligned} G'(t) &\geq \left[(1-\sigma) - \frac{\eta 2^{\alpha-1}\gamma^\alpha\epsilon\Gamma(-\alpha)}{B(p+1)^\alpha}\right]H^{-\sigma}(t)H'(t) + \left[1+\frac{\eta 2^{\alpha-2}\gamma^\alpha\epsilon^2\Gamma(-\alpha)}{B(p+1)^{\alpha-1}}H^{-\sigma}(t)\right]H(t) \\ &+ \left[\frac{A_1}{2} - \eta A_1\right]e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&+\epsilon\left[\frac{\eta 2^{\alpha-3}\gamma^{\alpha}A_1(p-1)\epsilon\Gamma(-\alpha)}{B(p+1)^{\alpha}}H^{-\sigma}(t)-(\eta\delta_4\gamma S_2^2+\delta_6 S_2^2)\right]e^{-\gamma\epsilon t}\|u(t)\|^2_{H^2_*(\Omega)}\\
&+\left[\frac{1}{2}+\eta\left(1-\frac{\gamma\epsilon}{4\delta_4}\right)-\frac{\epsilon}{4\delta_6}\right]e^{-\gamma\epsilon t}\|u_t(t)\|_2^2+\left[\eta-\frac{1}{p+1}-\frac{\eta\mu BC_2}{(p+1)^{\sigma}\Gamma(-\alpha)}\right]e^{-\gamma\epsilon t}\int_{\Omega}|u|^{p+1}dxdy \qquad (44)\\
&+\left(\lambda-\frac{\eta\mu BC_2}{(p+1)^{\sigma}\Gamma(-\alpha)}\right)+\theta F(t).
\end{aligned}
$$

Now, we choose are parameters carefully. First, recalling $\sigma=\frac{p-1}{2(p+1)}$ and selecting ϵ so small such that

$$
\epsilon\leq\epsilon_2:=\frac{1}{2}\frac{B(p+1)^{\alpha}(1-\sigma)}{\eta 2^{\alpha-1}\gamma^{\alpha}\Gamma(-\alpha)}, \tag{45}
$$

we see that the coefficient of the first term is positive. By choosing $\eta=\frac{p+3}{4(p+1)}$, $\delta_4=\delta_6=\frac{1}{2}$, and ϵ small enough so that

$$
\epsilon\leq\epsilon_3:=\frac{4(p-1)}{(p+11)^2S_2^2}, \tag{46}
$$

we find that the coefficient of $\|u(t)\|^2_{H^2_*(\Omega)}$ is positive. Next, we pick ϵ small enough such that

$$
\epsilon\leq\epsilon_4:=\frac{2(3p+5)}{(p+1)(p+11),} \tag{47}
$$

to get the coefficient of $\|u_t(t)\|_2^2$ greater or equal to $\frac{1}{2}$. We select B such that

$$
B<\frac{(p+1)^{\sigma}\Gamma(-\alpha)}{(p+3)\mu C_2}\min\left\{\frac{p-1}{2},4\lambda(p+1)\right\}, \tag{48}
$$

to see that the coefficient of $\int_{\Omega}|u|^{p+1}dxdy$ is greater than $\frac{p-1}{4(p+1)}$ and the term

$$
\left(\lambda-\frac{\eta\mu BC_2}{(p+1)^{\sigma}\Gamma(-\alpha)}\right)>0.
$$

Thus, for any ϵ positive small enough such that

$$
\epsilon<\min\{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\}, \tag{49}
$$

we arrive at

$$
G'(t)\geq H(t)+\frac{1}{2}\|u_t(t)\|_2^2+\frac{p-1}{4(p+1)}\int_{\Omega}|u|^{p+1}dxdy\;\forall\;t\geq 0. \tag{50}
$$

Using Cauchy-Schwarz and Young's inequalities, we have

$$
\begin{aligned}
|(u,u_t)_2|^{\frac{1}{1-\sigma}}&\leq\|u(t)\|_2^{\frac{1}{1-\sigma}}\|u_t(t)\|_2^{\frac{1}{1-\sigma}}\\
&\leq C_2\|u(t)\|_{p+1}^{\frac{1}{1-\sigma}}\|u_t(t)\|_2^{\frac{1}{1-\sigma}}\\
&\leq C_3\left(\|u(t)\|_{p+1}^{\frac{r_1}{1-\sigma}}+\|u_t(t)\|_2^{\frac{r_2}{1-\sigma}}\right),
\end{aligned} \tag{51}
$$

where $C_2=C_2(|\Omega|,p)>0,\ C_3=C_3(|\Omega|,p,\sigma)>0$ are constants and $\frac{1}{r_1}+\frac{1}{r_2}=1$. We recall that $\sigma=\frac{p-1}{2(p+1)}$, therefore we select $r_1=\frac{2(1-\sigma)}{1-2\sigma},\ \ r_2=2(1-\sigma)$, and arrive at

$$
|(u,u_t)_2|^{\frac{1}{1-\sigma}}\leq C_3\left(\|u(t)\|_{p+1}^{\frac{2}{1-2\sigma}}+\|u_t(t)\|_2^2\right). \tag{52}
$$

We observe that $\frac{2}{(p+1)(1-2\sigma)}=1$, so

$$\|u(t)\|_{p+1}^{\frac{2}{1-2\sigma}} = \int_{\Omega} |u|^{p+1} dxdy.$$

From the definition of $G(t)$, we have

$$\begin{aligned} G(t)^{\frac{1}{1-\sigma}} &= \left(H^{1-\sigma}(t) + \eta e^{-\gamma\epsilon t}(u,u_t)_2\right)^{\frac{1}{1-\sigma}} \leq 2^{\frac{1}{1-\sigma}}\left(H(t) + \eta^{\frac{1}{1-\sigma}} |(u,u_t)_2|^{\frac{1}{1-\sigma}}\right) \\ &\leq 2^{\frac{1}{1-\sigma}}\left(H(t) + C_3\eta^{\frac{1}{1-\sigma}}\left(\|u(t)\|_{p+1}^{\frac{2}{1-2\sigma}} + \|u_t(t)\|_2^2\right)\right) \\ &= 2^{\frac{1}{1-\sigma}}\left(H(t) + C_3\eta^{\frac{1}{1-\sigma}}\left(\int_{\Omega} |u|^{p+1}dxdy + \|u_t(t)\|_2^2\right)\right) \\ &\leq C\left(H(t) + \frac{1}{2}\|u_t(t)\|_2^2 + \frac{p-1}{4(p+1)}\int_{\Omega}|u|^{p+1}dxdy\right), \end{aligned} \tag{53}$$

for some positive constant C such that

$$C \geq 2^{\frac{1}{1-\sigma}} \max\left\{1, 2C_3\eta^{\frac{1}{1-\sigma}}, C_3\eta^{\frac{1}{1-\sigma}}\frac{4(p+1)}{p-1}\right\}.$$

A combination of (50) and (53) leads to

$$(G(t))^{\frac{1}{1-\sigma}} \leq CG'(t),\ \forall t \geq 0. \tag{54}$$

From (50), we see clearly that $G'(t) \geq 0$. It follows from the definition of $G(t)$ and the assumption on u_0 and u_1 that

$$G(t) \geq G(0) > \eta(u_0,u_1)_2 \geq 0. \tag{55}$$

Hence, $G(t) > 0$. Integrating (54) over $(0,t)$ yields

$$(G(t))^{\frac{-\sigma}{1-\sigma}} \leq (G(0))^{\frac{-\sigma}{1-\sigma}} - \frac{\sigma}{C(1-\sigma)}t$$

which gives

$$(G(t))^{\frac{\sigma}{1-\sigma}} \geq \frac{1}{(G(0))^{\frac{-\sigma}{1-\sigma}} - \frac{\sigma}{C(1-\sigma)}t}. \tag{56}$$

From (56), we obtain that $G(t)$ blows up in time

$$T^* \leq \frac{C(1-\sigma)}{\sigma\,(G(0))^{\frac{\sigma}{1-\sigma}}}. \tag{57}$$

This completes the proof. □

5. Conclusion

In this paper, we have studied a plate equation supplemented with partially hinged boundary conditions as model for suspension bridge in the presence of fractional damping and non-linear source terms. We showed that the solution blows up in finite time. We saw that, even in the present of a weaker damping, the bridge will collapse in infinite time when the power p of the non-linear source term is sufficiently large. This is a very important factor for engineers to consider when constructing such types of bridges.

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