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On the Riemann Hypothesis

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Short Research Article

Abstract

This paper provides an exact characterisation of the zeroes of the Riemann zeta function. The characterisation is based on a theorem about random vectors, which says that under some conditions, if a vector is always in the convex hull of the conditional expectations corresponding to any two mutually exclusive and exhaustive events, then the unconditional expectation of the random vector is equal to that vector.

Keywords: Riemann zeta function; conditional expectations; law of total probability; Riemann Hypothesis.

2010 Mathematics Subject Classification: 30A99, 30E99.

1 Introduction

In this paper, we show a characterisation of the zeroes of the Riemann zeta function ([1],[2]). The characterisation is fairly abstract and proves a known result regarding the zero free region, for which the prime factor characterisation of the Riemann zeta function yields proof. Our methods differ given our expectations based characterisation and further applies also to the critical region. Hence, this may be applied to the study of zeroes, possibly allowing us to prove propositions concerning the Riemann Hypothesis. Another result in the paper allows us to study topological properties, suggesting observations that would be consistent with the Riemann Hypothesis i.e. nowhere denseness of the set of zeroes. Prior propositions and dissimilar perspectives concerning this problem

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may be found in [3], [4], [5], [6] and [7]. Lastly, we find that the behaviour of the sequence $\{\ln(n)\}_{n\in\mathbb{Z}^+}$ as studied here has a curious similarity (though unrelated) with the study of uniform distribution of sequences ([8]).

2 Riemann Hypothesis

We study the Riemann zeta function ([9],[2]), by defining it explicitly as a function on a subset $S \subseteq \mathbb{R}^2$, mapping into $\mathbb{R}^2$. Namely, we define the set S as

$$S = \{(\sigma, t) \in \mathbb{R}^2 : \sigma \in (0, 1); t \neq 0\} \tag{1}$$

corresponding to the space of all complex numbers, whose real part is in $(0, 1)$. The space of all complex numbers is itself, of course, identified with the space $\mathbb{R}^2$. For any given $s = (\sigma, t) \in \mathbb{R}^2$, we interpret σ as the real part and t as the imaginary part.

We now recall some definitions about complex numbers regarding addition, multiplication and exponentiation of complex numbers ([10],[2]). Addition is defined simply as vector addition in $\mathbb{R}^2$ i.e. if we have points $s = (\sigma, t)$ and $s' = (\sigma', t')$ in $\mathbb{R}^2$, then $s + s' = (\sigma + \sigma', t + t')$. Multiplication, as defined, incorporates the assimilation of the arithmetic of real numbers with i, using the fact that $i^2 = -1$. Hence, $s \times s' := (\sigma\sigma' - tt', \sigma t' + t\sigma')$. Exponentiation, by applying Euler's formula ([2]), gives the following expression

$$e^s := e^\sigma(\cos(t), \sin(t)), \tag{2}$$

in which $e^s \in \mathbb{R}^2$ is the vector defined by multiplying the scalar e^σ with the vector $(\cos(t), \sin(t)) \in \mathbb{R}^2$. Further, $(\cos(t), \sin(t))$ also belongs to the unit circle in $\mathbb{R}^2$. By now applying this expression for exponentiation, we obtain that for any positive integer $n \geq 1$, and $s = (\sigma, t)$, we have that

$$\frac{1}{n^s} = e^{-\sigma \ln(n)}(\cos(-t \ln(n)), \sin(-t \ln(n))). \tag{3}$$

2.1 The Riemann Hypothesis and the expectation of a random vector

The Riemann zeta function is defined as the function $\zeta : S \to \mathbb{R}^2$,

$$\zeta(s) := \sum_{n=1}^{\infty}(1 - \frac{1}{2^{1-s}}) \times \frac{(-1)^{n+1}}{n^s}. \tag{4}$$

Now, we think of the set of all positive integers $\mathbb{Z}^+$ as a measurable space ([11],[12]) with the power set sigma-field i.e. $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$. For any point $s \in \mathbb{R}^2$, we define the measurable function $X_s : \mathbb{Z}^+ \to \mathbb{R}^2$ and measure $\mu_s : 2^{\mathbb{Z}^+} \to [0, \infty]$ as

$$X_s(n) := (-1)^{n+1}\Big[(1 - \frac{1}{2^{1-s}}) \times (\cos(-t \ln(n)), \sin(-t \ln(n)))\Big], \tag{5}$$

and

$$\mu_s(E) := \sum_{n\in E} e^{-\sigma \ln(n)} \tag{6}$$

respectively.

Hence, it is apparent from the definition of X_s and μ_s above, that $\zeta(s) = \mathbb{E}_{\mu_s}[X_s]$ and that hence $\zeta(s) \neq 0$ if and only if $\mathbb{E}_{\mu_s}[X_s] \neq 0$, denoting $\mathbb{E}_{\mu_s}$ as the expectation operator corresponding to the measure μ_s, defined in this context as follows : for any measure μ, the operator $\mathbb{E}_\mu$ is defined as any linear function ([11],[13]) $\mathbb{E}_\mu : l^\infty \times l^\infty \to \mathbb{R}^2$ such that

$$\mathbb{E}_\mu(X) = \sum_{n\in\mathbb{Z}^+} \mu(\{n\})X(n), \tag{7}$$

if it is true that the partial sums $\sum_{n=1}^{m} \mu(\{n\})X(n)$ converge for the given $X \in l^\infty \times l^\infty$. For a subset $E \subseteq \mathbb{Z}^+$, the expectation over the set E is given by $\mathbb{E}_\mu[\mathbb{I}_E X]$.

We first prove a proposition that would lead to the characterisation. The proposition is about convergent infinite series in $\mathbb{R}^d$. For points $s, s' \in \mathbb{R}^d$, we define the line joining the points as $< s, s' > := \{\theta s + (1-\theta)s' : \theta \in \mathbb{R}\}$ and define $< s > := < s, 0 >$.

Proposition 2.1. *Suppose that $\{x_n\}_{n=1} \subseteq \mathbb{R}^d$ such that there exist $n, m \in \mathbb{Z}^+$ with*

$$x_m \notin < x_n > . \tag{8}$$

Further, suppose that $\sum_{n\in\mathbb{Z}+} x_n$ exists. Then, we have that $\sum_{n\in\mathbb{Z}+} x_n \neq 0$ if and only if there exists $m \in \mathbb{Z}^+$ such that $0 \notin < \sum_{n=1}^{m} x_n, \sum_{n=m+1}^{\infty} x_n >$.

Proof. We first prove the "if" part of the proposition. Suppose that it is true that there exists $m \in \mathbb{N}$ such that $0 \notin < \sum_{n=1}^{m} x_n, \sum_{n=m+1}^{\infty} x_n >$. Suppose, for contradiction $\sum_{n\in\mathbb{Z}+} x_n = 0$. Then, we have

$$\frac{1}{2}\sum_{n=1}^{m} x_n + \frac{1}{2}\sum_{n=m+1}^{\infty} x_n = 0. \tag{9}$$

Hence, $0 \in < \sum_{n=1}^{m} x_n, \sum_{n=m+1}^{\infty} x_n >$, which is a contradiction.

We next prove the only if part of the proposition. Suppose that $\sum_{n\in\mathbb{Z}+} x_n \neq 0$. Define,

$$m := \min\{m' : x_{m'} \notin < \sum_{n\in\mathbb{Z}+} x_n >\}. \tag{10}$$

The condition that is given by 8 shows that m is well-defined. By definition, we have for m that

$$\sum_{n=1}^{m} x_n \notin < \sum_{n\in\mathbb{Z}+} x_n > . \tag{11}$$

This also means that $\sum_{n=1}^{m} x_n \neq 0$ and hence from the above conclusion, we have that $\sum_{n=m+1}^{\infty} x_n \neq 0$ and that

$$\sum_{n\in\mathbb{Z}+} x_n \notin < \sum_{n=1}^{m} x_n > . \tag{12}$$

Suppose, for contradiction that $0 \in < \sum_{n=1}^{m} x_n, \sum_{n=m+1}^{\infty} x_n >$. Then, we have that $\sum_{n=m+1}^{\infty} x_n \in < \sum_{n=1}^{m} x_n >$. This would then mean that

$$\sum_{n=1}^{m} x_n + \sum_{n=m+1}^{\infty} x_n \in < \sum_{n=1}^{m} x_n >, \tag{13}$$

which contradicts 12. Hence, $0 \notin < \sum_{n=1}^{m} x_n, \sum_{n=m+1}^{\infty} x_n >$. □

We prove the following main proposition about the function ζ.

Proposition 2.2. *Suppose $s \in S$. Then, $\zeta(s) = 0$ if and only if for each subset $E \subseteq \mathbb{Z}^+$, we have that $0 \in < \mathbb{E}_{\mu_s}[\mathbb{I}_E X_s], \mathbb{E}_{\mu_s}[\mathbb{I}_{\mathbb{Z}^+ \setminus E} X_s] >$.*

Proof. The proof follows from Proposition 2.1. □

We state an application of the above proposition. For each $m \in \mathbb{Z}_+$, define the functions $\zeta_m : S \to \mathbb{R}^2$ and $\zeta_{-m} : S \to \mathbb{R}^2$ as

$$\zeta_m(s) := \sum_{n=1}^{m}(1 - \frac{1}{2^{1-s}}) \times \frac{(-1)^{n+1}}{n^s}, \tag{14}$$

$$\zeta_{-m}(s) := \sum_{n=m+1}^{\infty}(1 - \frac{1}{2^{1-s}}) \times \frac{(-1)^{n+1}}{n^s}. \tag{15}$$

Hence, for each $s \in S$, we have the vectors $\zeta_m(s) = (\zeta_{m,1}(s), \zeta_{m,2}(s)) \in \mathbb{R}^2$ and $\zeta_{-m}(s) = (\zeta_{-m,1}(s), \zeta_{-m,2}(s)) \in \mathbb{R}^2$. By applying Heron's formula ([14]), one may define the function $F_m : S \to \mathbb{R}$ as

$$\begin{aligned} F_m(s) &:= 4(\zeta_{m,1}^2(s) + \zeta_{m,2}^2(s))(\zeta_{-m,1}^2(s) + \zeta_{-m,2}^2(s)) - (\zeta_{m,1}^2(s) + \zeta_{m,2}^2(s) \\ &+\zeta_{-m,1}^2(s) + \zeta_{-m,2}^2(s) - (\zeta_{m,1}(s) - \zeta_{-m,1}(s))^2 - (\zeta_{m,2}(s) - \zeta_{-m,2}(s))^2)^2. \end{aligned} \tag{16}$$

From Proposition 2.2, we have that $\zeta(s) = 0$, if and only if $F_m(s) = 0$, for each $m \in \mathbb{Z}_+$. Hence, since $F_m(s) \geq 0$, we have that if $\zeta(s) = 0$, then $s \in \arg\min_{s' \in S} F_m(s')$, for each $m \in \mathbb{Z}_+$. Hence, if $\zeta(s) = 0$, one obtains that $\nabla_{s'} F_m(s) = 0$, for each $m \in \mathbb{Z}_+$.

The expectation operator defined previously also leads to another characterisation of the zeroes of the Riemann zeta function. Let $\mathfrak{Z} = \{s \in S : \zeta(s) = 0\}$ and $\mathfrak{M} = \{(X, \mu) \in l^\infty \times l^\infty : \mathbb{E}_\mu(X) = 0\}$. Then, $\mathfrak{M}$ is nowhere dense ([15]) in $l^\infty \times l^\infty$, with the topology of pointwise convergence. Further, the map $f : S \to l^\infty \times l^\infty$ defined by $f(s) = (X_s, \mu_s)$ is continuous ([16]). The next proposition characterises the behaviour of the zeroes of the Riemann zeta function on compact subsets of S.

Proposition 2.3. *Let $S' \subseteq S$ be a compact subset of S. Suppose $f(S') \cap \mathfrak{M}$ is nowehere dense in the subspace topology generated by $f(S')$. Then, $S' \cap \mathfrak{Z}$ is nowhere dense in S'. Hence, if ψ is a category measure on S', it follows that $\psi(S' \cap \mathfrak{Z}) = 0$.*

Proof. By definition of f, we have that $f^{-1}(f(S') \cap \mathfrak{M}) = S' \cap \mathfrak{Z}$. Suppose for contradiction, $S' \cap \mathfrak{Z}$ is not nowhere dense. Then, this means that $\text{int}(\text{cl}(S' \cap \mathfrak{Z})) \neq \emptyset$. Hence, $\text{int}(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M}))) \neq \emptyset$. Since S' is compact, the map $f|_{S'}$ is an open map from S' to $f(S')$, with the subspace topology on $f(S')$. Hence, the set $f(\text{int}(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M}))))$ is an open set in $f(S')$. However, observe that $f(\text{int}(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M})))) \subseteq f(\text{cl}(f^{-1}(f(S') \cap \mathfrak{M}))) \subseteq f(f^{-1}(\text{cl}(f(S') \cap \mathfrak{M}))) \subseteq \text{cl}(f(S') \cap \mathfrak{M})$ (the second inclusion follows since f is continuous). This is a contradiction, since we have that $\text{int}(\text{cl}(f(S') \cap \mathfrak{M})) = \emptyset$. ☐

The Riemann zeta function, for values $\sigma > 1$, is given by the infinite sum

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}. \tag{17}$$

In the next proposition, applying the perspectives presented above involving the expectations operator, we may show a proposition previously known, for the Riemann zeta function on this domain.

Proposition 2.4. *For $s = (\sigma, t)$ such that $\sigma \geq 2$, we have that $\zeta(s) \neq 0$.*

Proof. Define, on the set of positive integers, the probability measure $\mu_s(\{n\}) := \frac{(1/n)^\sigma}{\sum_{m \geq 1}(1/m)^\sigma}$ and the random variable $X_s(n) = (\cos(-t\ln(n)), \sin(-t\ln(n)))$. Note that

$$\begin{aligned} \mu_s(\{1\}) &= \frac{1}{1 + \sum_{m \geq 2}(1/m)^\sigma} && (18) \\ &\geq \frac{1}{1 + \sum_{m \geq 2}(1/m)^2} && (19) \\ &> \frac{1}{1 + \sum_{m \geq 1}(1/2)^m} && (20) \\ &= 1/2 && (21) \end{aligned}$$

for the value $n = 1$, and we have that $X(\{1\}) = (1, 0)$ as $\ln(1) = 0$. Suppose, for contradiction that $\zeta(s) = 0$. Then, for the set $E = \{1\}$, we have that $\mathbb{E}_{\mu_s}[X_s] = \mu_s(E)\mathbb{E}_{\mu_s}[X_s|E] + (1 - \mu_s(E))\mathbb{E}_{\mu_s}[X_s|\mathbb{Z}^+\backslash E] = 0$. Let $\alpha = \mu_s(E)$, $x = \mathbb{E}_{\mu_s}[X_s|E]$ and $z = \mathbb{E}_{\mu_s}[X_s|\mathbb{Z}^+\backslash E]$. Since $\alpha > 1/2$, $x = (1, 0)$ and $z = (z_1, 0)$ such that $z_1 \in [-1, 1]$, we have that $\alpha x + (1 - \alpha)z \neq 0$, which is a contradiction. □

3 Conclusion

In this paper, we proved a theorem that yields a characterisation of the zeroes of the Riemann zeta function based on observations about zero expectation random vectors. This presents a new angle to the problem of characterising zeroes and may offer new perspectives on the Riemann Hypothesis.

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Competing Interests

Author has declared that no competing interests exist.

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